Cornell University Library

BOUGHT WITH THE INCOME
FROM THE
SAGE ENDOWMENT FUND
THE GIFT OF
Henry W. Sage
1891

A.76435 17/11/94

CORNELL UNIVERSITY LIBRARY
3 1924 091 870 307

Cornell University
Library

The original of this book is in
the Cornell University Library.

There are no known copyright restrictions in
the United States on the use of the text.

http://www.archive.org/details/cu31924091870307

THEORETICAL MECHANICS

AN

ELEMENTARY TREATISE

ON

THEORETICAL MECHANICS

BY

ALEXANDER ZIWET
ASSISTANT PROFESSOR OF MATHEMATICS IN THE
UNIVERSITY OF MICHIGAN

PART III: KINETICS

New York
MACMILLAN AND CO.
AND LONDON
1894

All rights reserved

Copyright, 1894,
By MACMILLAN AND CO.

Norwood Press:
J. S. Cushing & Co. — Berwick & Smith.
Boston, Mass., U.S.A.

PREFACE.

About one-half of this volume is devoted to the kinetics of a particle, the remainder being given to the study of the kinetics of a rigid body and a brief discussion of the fundamental principles of the kinetics of a system.

The first part of the chapter on the motion of a particle (impact, rectilinear motion) gradually introduces and illustrates in an elementary way such fundamental ideas as momentum, impulse, kinetic energy, force, work, potential energy, power. Then the general equations of motion of a particle are discussed; and the principle of kinetic energy (or *vis viva*), that of angular momentum (or of areas), and the principle of d'Alembert are explained and applied, first to the motion of a free particle (central forces), then to constrained motion. The example of such recent writers as Budde and Appell has been followed in treating the constraints of a particle with more than usual fulness, introducing generalized co-ordinates, and establishing the equations of motion of a particle in the Lagrangian form. It is believed that this will materially aid the student in understanding the use of these methods in the general case of the motion of a system.

The chapter on the motion of a rigid body, after a discussion of the fundamental principles and of the theory of moments and ellipsoids of inertia, takes up separately the action of impulses and the motion under continuous forces. The last chapter, on the motion of a system, is necessarily brief, owing to the elementary character of the treatise. A sketch of the theory of Lagrange's generalized co-ordinates and of Hamilton's principle is, however, included.

For a shorter course, the Articles 104–159, 180–188, 190–217, 225, 262, 268–272, 274–290, 304–310, 320–323, 327, 329–332, 336–356, 391–397 may be omitted.

ALEXANDER ZIWET.

Ann Arbor, Mich.,
October, 1894.

CORRIGENDA TO PART II.

Page 34, line 2, for "uniformly" read "with constant acceleration."

Page 40, last line, for "$m\frac{dv}{ds}$" read "$mv\frac{dv}{ds}$."

Page 71, line 5, for "of axis x" read "as axis of x."

Page 74, line 1, for "$y + c$" read "$y + C$."

Page 82, line 4, for "by the area" read "by twice the area."

Page 101, line 10, for "$-$" read "$+$."

Page 105, line 3 from the bottom, interchange W_1 and W_3.

Page 144, line 7, for "P" read "π."

Page 151, line 7 from the bottom, for "$\frac{\pi}{x}$" read "$\frac{a}{x}$."

Page 160, Ex. (1), for "9·3" read "9·8."

Page 181, line 10, for "11, 12" read "90, 92."

Page 181, line 13, for "$\frac{5}{4}$" read "$\frac{5}{3}$."

Page 182, line 3, for "4 ft." read "$3\frac{1}{4}$ ft."

Page 182, lines 12, 13 should read as follows: "$C = 1$, $D = 1\frac{1}{3}$, $E = 6\frac{2}{3}$, $AB = 4·5$, $BC = 4·1$, $CD = 4·0$, $DE = 4·2$, $EF = 4·9$; reaction at $A = 2·8$, at $F = 7·2$."

Page 183, line 7, read "(1) (a) $\frac{\sin(\theta - \phi)}{\cos\phi} W < P < \frac{\sin(\theta + \phi)}{\cos\phi} W$."

Page 183, line 10, for "2 tons" read "3 tons."

CONTENTS.

CHAPTER V.

KINETICS OF A PARTICLE.

PAGE

I. IMPUSLES; IMPACT OF HOMOGENEOUS SPHERES 1

II. RECTILINEAR MOTION 18

III. FREE CURVILINEAR MOTION:

1. General principles 38
2. Central forces 56
3. The problem of two bodies 79

IV. CONSTRAINED MOTION:

1. Introduction 85
2. Motion on a fixed curve 87
3. Motion on a fixed surface 98
4. Motion on a moving or variable curve or surface . 104

V. LAGRANGE'S FORM OF THE EQUATIONS OF MOTION 113

CHAPTER VI.

KINETICS OF A RIGID BODY.

I. GENERAL PRINCIPLES 119

II. MOMENTS OF INERTIA AND PRINCIPAL AXES:

1. Introduction 132
2. Ellipsoids of inertia 139
3. Distribution of principal axes in space 150

PAGE
III. RIGID BODY WITH A FIXED AXIS 157
IV. RIGID BODY WITH A FIXED POINT:
1. Initial motion due to impulses 169
2. Continuous motion under any forces 176
3. Continuous motion without forces 189
V. FREE RIGID BODY:
1. Initial motion due to impulses 196
2. Continuous motion 207

CHAPTER VII.

MOTION OF A VARIABLE SYSTEM.

I. FREE SYSTEM 210
II. SYSTEM SUBJECT TO CONDITIONS 214

ANSWERS 227

THEORETICAL MECHANICS.

CHAPTER V.

KINETICS OF A PARTICLE.

I. *Impulses; Impact of Homogeneous Spheres.*

1. Momentum and Impulse. A particle of mass m, moving with the velocity v, is said to have the *momentum* mv (see Part II., Art. 56). As long as this momentum remains constant, the particle will move in a straight line with constant velocity v (Newton's first law of motion, Part II., Art. 74). Any change occurring in the momentum is ascribed to the action of a force F on the particle.

2. If the rate of change of momentum is constant during the time $t'-t$, the force F is constant, and is measured by the change of momentum in the unit of time; that is,

$$F(t'-t)=mv'-mv, \tag{1}$$

where v is the velocity at the time t, and v' the velocity at the time t' (Newton's second law of motion). As the product $F(t'-t)$ of a constant force into the time during which it acts is called the *impulse* of the force during this time (Part II., Art. 61), equation (1) can be expressed in words by saying that *the impulse of the force is equal to the change of momentum.*

This proposition is easily seen to hold even for a variable force. For such a force, we have

$$Fdt=d(mv);$$

hence, by integration,

$$\int_t^{t'} F dt = mv' - mv, \tag{2}$$

where the time-integral in the left-hand member is the impulse of the variable force F during the time $t' - t$.

3. It appears, from equations (1) and (2), that a very large force may produce a finite change of momentum in a very short interval of time, but that it would require an infinite force to produce an *instantaneous* change of momentum of finite amount. The impact of one billiard ball on another, the blow of a hammer, the stroke of the ram of a pile-driver, the shock imparted by a falling body, by a projectile, by a railway train in motion, by the explosion of the powder in a gun, are familiar instances of large forces acting for only a very short time and yet producing a very appreciable change of velocity. The time of action, $t' - t$, of such a force is the very brief period during which the colliding bodies are in contact. The force, F, is a pressure or an elastic stress exerted by either body on the other during this time.

Forces of this kind are called *impulsive*, or *instantaneous, forces*.

4. In the case of such impulsive forces, it is generally difficult or impossible by direct observation or experiment to determine separately the very brief time of action, $t' - t$, as well as the magnitude F of the impulsive force. Moreover, what is of most practical importance and interest in such cases of impact is, generally, not the force itself, but the change of momentum produced, *i.e.* the impulse of the impulsive force.

In the present section, which is devoted to the study of the simplest cases of impact, we shall therefore deal with impulses and momenta, and not with forces.

5. It should be observed that many authors use the name *impulsive*, or *instantaneous, force* for what has here been called the impulse of the impulsive force. They define an impulsive force as the limiting value of the integral $\int_t^{t'} Fdt$ when F increases indefinitely, while at the same time the difference of the limits, $t' - t$, is indefinitely diminished; in other words, as the impulse of an infinite force producing a finite change of momentum in an infinitesimal time.

According to this definition, an impulsive or instantaneous force is a magnitude of a character different from that of an ordinary force, and is measured by a different unit. Its dimensions are $\mathbf{MLT}^{-1}$, and not $\mathbf{MLT}^{-2}$. Its unit is the same as the unit of momentum. Indeed, it is not a force, but an impulse.

We can arrive at this idea of an instantaneous force from a somewhat different point of view. Just as in kinematics (Part I., Arts. 104 and 156) we may distinguish accelerations of different orders, regarding velocity as acceleration of order zero, so in dynamics instantaneous forces may be regarded as forces of order 0, ordinary (continuous) forces as forces of order 1, the product of mass into the acceleration of the second order as a force of the second order, and so on.

In the present elementary treatise, no use is made of these considerations. The word force is always used as meaning the product of mass into acceleration of the first order, and the time-integral $\int_t^{t'} Fdt$ is always called impulse, and not impulsive force.

6. The momentum mv of a particle P of mass m, moving with the velocity v, can be represented geometrically by a vector (more exactly by a localized vector, or rotor), *i.e.* by a segment of a straight line drawn through P and representing by its length the magnitude of the momentum, by its direction and sense the direction and sense of the velocity. Hence, *the composition and resolution of momenta follows the same rules that hold for forces.*

7. Let us consider two particles P_1, P_2 of masses m_1, m_2, having equal and parallel velocities v, and let their momenta m_1v, m_2v be represented by their vectors (Fig. 1). The two particles may be regarded as forming a single moving system; as the velocities are equal in magnitude, direction, and sense, the system has a motion of translation. According to the rule for compounding parallel rotors (explained for rotors representing angular velocities in Part I., Arts. 253–255, and for rotors representing forces in Part II., Arts. 104–107), the resultant momentum is parallel to the given momenta and equal in magnitude to their algebraic sum $(m_1 + m_2)\,v$; and its line divides the distance P_1P_2 in the inverse ratio of the momenta m_1v, m_2v, or of the masses m_1, m_2. *The resultant passes, therefore, through the centroid* P *of the masses* m_1, m_2.

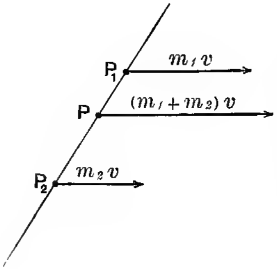

Fig. 1.

8. It is easy to see how this proposition can be generalized. If any number of particles, all having equal and parallel velocities, be given, the resultant momentum, or *the momentum of the system*, is equal to the mass of the system multiplied by the common velocity, and passes through the centroid of the system.

Thus, in the case of a rigid body having a velocity of translation v, but no rotation, the whole mass M of the body may be regarded as concentrated at the centroid, and the momentum of the centroid, Mv, is then equal to that of the body.

9. But we can speak of the momentum of a system of particles even when their velocities are not of equal magnitude but only parallel.

Let x be the distance, at the time t, of any particle P of mass m from some fixed plane, which, for the sake of simplicity, we may take at right angles to the direction of the velocity. Then the distance $\bar{x}$ of the centroid G of the system at the time t from the same plane is (Part II., Art. 13)

$$\bar{x} = \frac{\Sigma mx}{\Sigma m} = \frac{\Sigma mx}{M}. \tag{3}$$

Differentiating this equation with respect to the time, and remembering that $dx/dt=v$ is the velocity of the particle P, we find for the velocity $d\bar{x}/dt=\bar{v}$ of the centroid

$$\bar{v}=\frac{\Sigma mv}{M}. \tag{4}$$

10. In the special case of two particles P_1, P_2 of masses m_1, m_2, moving with the velocities v_1, v_2 in the same straight line, we have

$$\bar{v}=\frac{m_1v_1+m_2v_2}{m_1+m_2}. \tag{5}$$

If the velocities v_1, v_2 be constant, this equation shows that the centroid moves with constant velocity and constant momentum in the same line.

Similarly, the more general equation (4) of the preceding article,

$$M\bar{v}=\Sigma mv, \tag{6}$$

shows that *the momentum of a system of particles moving with constant velocities in the same direction remains constant, i.e.* the centroid of such a system moves with constant velocity in a straight line. It is to be noticed that the velocities need not be all of the same sense; that is, v may be positive for some particles and negative for others.

This proposition may be regarded as a generalization of Newton's first law of motion.

11. Direct Impact. We proceed to consider the particular case of two homogeneous spheres of masses m, m', whose centres C, C' move with velocities u, u' in the same straight line. The spheres are supposed not to rotate but to have a motion of pure translation; then their momenta are mu, $m'u'$, and can be represented by two vectors drawn from the centres C, C' along the line CC' (Fig. 2). To fix the ideas we assume the velocities

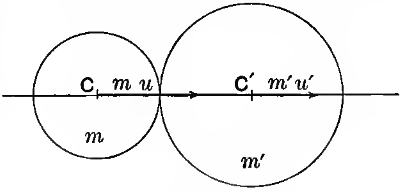

Fig. 2.

u, u' to have the same sense and $u>u'$, so that m will finally impinge upon m'. The case when the velocities are of opposite sense will not require special investigation, as only the sign of u' would have to be changed.

It is our object to determine the velocities v, v′ *of* m, m′, *immediately after impact*, when the velocities u, u' immediately before impact are given.

The results here derived for homogeneous spheres hold, generally, whatever the shape of the impinging bodies, provided that they do not rotate, and that the common normal at the point of contact passes through both centroids.

12. If the spheres were perfectly rigid, the problem would be indeterminate, for there is no way of deciding how the velocities would be affected by the collision.

Natural bodies are not perfectly rigid. The effect of the impact will, in general, consist in a compression of the portions of the bodies brought into contact. Moreover, all natural bodies possess a certain degree of elasticity; the compression will therefore be followed by an extension, each sphere tending to regain its shape at least partially.

The compression acts as a retarding force on the impinging sphere m, and as an accelerating force on m'. It will last until the velocities u, u' have become equal, say $=w$. During the subsequent period of extension, or *restitution*, the elastic stress still further diminishes the velocity of m, and increases that of m', until they become, say, v, v'.

13. The stress varies, of course, during the whole time τ of compression and restitution. But, according to Newton's third law, the pressure F exerted at any instant by m on m' must be equal and opposite to the pressure F' exerted by m' on m at the same instant. Since $F=m\,du/dt$, $F'=m'du'/dt$, and $F=-F'$ at any instant during the time τ, we have

$$\int_0^\tau F dt = -\int_0^\tau F' dt, \text{ or } m\int_0^\tau du = -m'\int_0^\tau du',$$

whence $$mv - mu = -(m'v' - m'u'),$$

or $$mv + m'v' = mu + m'u'; \tag{7}$$

that is, *the total momentum after impact is equal to that before impact.*

14. This proposition will evidently hold for any number of spheres whose centres move in the same line, and can then be expressed in the form

$$\Sigma mv = \Sigma mu. \tag{8}$$

It can be regarded as a special case of the so-called principle of the conservation of the motion of the centroid to be proved hereafter for any system not acted upon by external forces. On the other hand, the proposition can be looked upon as a further generalization of Newton's first law of motion. While the latter asserts that the momentum of a *particle* remains unchanged as long as no external forces act upon it, our law of impact asserts the same thing for the momentum of a *system*.

15. If the spheres were *perfectly non-elastic*, there would be only compression and no subsequent extension. As at the end of the period of compression, the velocities u, u' have both become equal, viz. $= w$ (Art. 12), the spheres after impact would have the common velocity

$$w = \frac{mu + m'u'}{m + m'}. \tag{9}$$

16. If the spheres were *perfectly elastic*, *i.e.* if the elastic stress following the compression, or the so-called *force of restitution*, were just equal to the preceding stress of compression, the spheres would completely regain their original shape. In this case, the elastic stress causes the impinging sphere m to lose during the period of restitution an amount of momentum $m(w - u)$ equal to that lost during the period of compression. Hence, the final velocity of m after impact would be

$$v = w - (u - w) = 2w - u.$$

Similarly, we have for the other sphere m'

$$v' = w + (w - u') = 2w - u'.$$

As w is known from (9), the velocities after impact can be determined by means of these formulæ for perfectly elastic spheres.

17. In general, physical bodies are *imperfectly elastic*, the force of restitution being less than that of the original compression; that is, we have

$$(w - v) = e(u - w),$$
$$(v' - w) = e(w - u'),$$

where e is a proper fraction whose limiting values are 0 for perfectly inelastic bodies and 1 for perfectly elastic bodies. This fraction e, whose value for different materials must be determined experimentally, is called the *coefficient of restitution* (or less properly, the coefficient of elasticity).

18. To eliminate w we have only to add the last two equations; this gives

$$v' - v = e(u - u'); \tag{10}$$

that is, *the ratio of the relative velocity after impact to the relative velocity before impact is constant and equal to the coefficient of restitution.*

This proposition, in connection with the proposition of Art. 13, expressed by formula (7), is sufficient to solve all problems of so-called *direct* impact, *i.e.* when the centres of the spheres move in the same line.

19. As the coefficient e is frequently difficult to determine, the limiting cases $e = 0$, $e = 1$ are important as giving approximate solutions for certain classes of substances.

Thus, for nearly inelastic bodies (such as clay, lead, etc.) we may put $e = 0$, whence, by (10), $v' = v$, *i.e.* the velocities of the spheres become equal after impact; and the value of the common velocity is found from (7) as

$$v = \frac{mu + m'u'}{m + m'},$$

which agrees with the result (9) found in Art. 15. For perfectly elastic bodies $e=1$, and formula (10) shows that in this case the relative velocity after impact is numerically equal to that before impact, but of opposite sense.

20. Exercises.

(1) Two balls of clay ($e=0$) weighing 2 and 3 oz. move in the same direction. The heavier ball impinges from behind upon the lighter ball at the moment when the latter moves at the rate of 15 ft. per second. If the velocity of the lighter ball is doubled by the impact, what was the original velocity of the heavier ball?

(2) Two glass balls ($e=1$) weighing 1 lb. and 12 oz., respectively, move in the same line with velocities of 5 and 4 ft. per second. What are their velocities after impact (*a*) if their original velocities were of the same sense, (*b*) if they were of opposite sense?

(3) A ball weighing 5 lbs., while moving with a speed of 51 ft. per second, overtakes a ball of 7 lbs. moving in the same line at the rate of 40 ft. per second. If the coefficient of restitution be $\frac{1}{3}$, what are the velocities of the two balls after impact?

(4) With the data of Ex. (3), show that the velocities after impact would be equal if the balls were perfectly inelastic, and that these velocities would differ more than in Ex. (3) if the balls were perfectly elastic.

(5) Find the velocity with which an elastic ball rebounds from a fixed surface after impinging upon it perpendicularly with a velocity u.

(6) To determine the coefficient of restitution, a ball is dropped from a height H on a fixed horizontal plate of the same material, and the height of rebound h is measured. Show that $e=\sqrt{h/H}$.

(7) A ball is dropped from a height $H=12$ ft. on a fixed horizontal plate. Find the height h to which it will rebound if $e=\frac{5}{6}$.

(8) If not disturbed, the ball in Ex. (7) will continue to fall and rebound alternately. (*a*) What height does it reach at the tenth rebound? (*b*) In what time does it come to rest? (*c*) What is the whole space described?

(9) A number of equal, perfectly elastic balls are placed in contact so that their centres are in a straight line. An equal ball impinges with a velocity u along this line on the first ball of the row. Show that the

last ball of the row will move off with the velocity u, while all the other balls will remain at rest.

(10) Find the velocity of the last (nth) ball in Ex. (9), when the coefficient of restitution is e.

(11) An inelastic ball of 8 lbs. is moving with a velocity of 12 ft. per second. (*a*) With what velocity must a ball of 24 lbs. meet it to arrest its motion? (*b*) With what velocity would the ball of 24 lbs. have to impinge from behind on the ball of 8 lbs. to double its velocity?

(12) A ball m impinges upon a ball m' from behind with a velocity u. Determine the velocities after impact, both for inelastic and for perfectly elastic balls: (*a*) when m' is originally at rest; (*b*) when m' is at rest and very large in comparison with m; (*c*) when m' has the initial velocity u', and is very large in comparison with m.

21. Kinetic Energy. A particle of mass m, moving with the velocity v, has the kinetic energy $\frac{1}{2}mv^2$ (Part II., Art. 71). As this is not a vector-quantity, the kinetic energy of a system consisting of any number of free particles is simply the algebraic sum, $\Sigma\frac{1}{2}mv^2$, of the kinetic energies of these particles. It is an essentially positive quantity, provided the masses are all positive.

The kinetic energy of a rigid body having a motion of pure translation is evidently $=\frac{1}{2}mv^2$, if m be the mass of the body and v the common velocity of all its points.

22. Change of kinetic energy is brought about by the action of force, and we have (Part II., Art. 72) for a constant force F,

$$\frac{1}{2}mv'^2-\frac{1}{2}mv^2=F(s'-s); \tag{11}$$

and for a variable force F,

$$\frac{1}{2}mv'^2-\frac{1}{2}mv^2=\int_s^{s'}Fds, \tag{12}$$

where the quantity in the right-hand member is called the *work* of the force. Thus a particle of mass m, falling from rest through a distance s, acquires its kinetic energy owing to the work done upon it by the constant attractive force, $F=mg$, of the earth, and we have

$$\tfrac{1}{2}mv^2=Fs=mgs.$$

The kinetic energy $\frac{1}{2}mv^2$, possessed by a particle of mass m, moving with the velocity v, can therefore always be regarded as equivalent to a certain amount of work. If the motion of this particle be *opposed* by a constant force or resistance F, the distance s through which it will go on moving until it comes to rest is of course determined from the same equation,

$$\tfrac{1}{2}mv^2=Fs. \tag{13}$$

It is then said that the kinetic energy of the particle is spent in overcoming the resistance F, or in doing work against the force F (see Part II., Art. 231).

23. In the case of direct impact of spheres, as considered in Art. 11, the velocity, and hence also the kinetic energy, of each sphere is in general changed by the impact; a transfer of kinetic energy can be said to take place. Thus, when a sphere at rest is struck by a moving sphere, kinetic energy is imparted to the former by the impulsive force, and this energy can then be spent in doing work against a resistance. Impact is therefore frequently used for the purpose of performing useful work.

24. For instance, to drive a nail into a wooden plank, the resistance F of the wood must be overcome through a certain distance s. This might be done by applying a pressure equal to F; as, however, this pressure would have to be very large, it is more convenient to impart to the nail, by striking it with a hammer, an amount of kinetic energy, $\frac{1}{2}mv^2$, equivalent to the work Fs that is to be done. Neglecting elasticity, and denoting the mass of the hammer by m, that of the nail by m', the velocity of the hammer at the moment when it strikes the head of the nail by u, we have, by (7),

$$mv+m'v'=mu,$$

or since, by (10), for inelastic impact $v' = v$,

$$v = \frac{m}{m + m'} u.$$

This is the common velocity of hammer and nail after the stroke. We find, therefore, by (13),

$$\tfrac{1}{2}(m + m') \cdot \left(\frac{mu}{m + m'}\right)^2 = Fs,$$

or

$$\frac{m}{m + m'} \cdot \tfrac{1}{2} mu^2 = Fs. \qquad (14)$$

25. It will be noticed that while the total kinetic energy of hammer and nail just before striking was $\frac{1}{2} mu^2 + 0$, the kinetic energy utilized for driving the nail is only the fraction $m/(m + m')$ of this total kinetic energy. The remaining portion of the original kinetic energy, viz.

$$\frac{m'}{m + m'} \cdot \tfrac{1}{2} mu^2, \qquad (15)$$

must be regarded as spent in producing the slight deformations of hammer and nail and such accompanying phenomena as vibrations of the plank, sound, heat, etc. For it is an experimental result of modern physical research that, wherever kinetic energy disappears as such, there is done an exactly equivalent amount of work. The apparently disappearing kinetic energy may either be transferred to some other body, as in the case of the vibrations of the plank, or it may reappear in the form of molecular vibrations, causing sound or heat; or it may be transformed into an equivalent amount of so-called potential energy. This physical fact is known as *the principle of the conservation of energy*.

26. In our example the total original kinetic energy, $\frac{1}{2} mu^2$, resolves itself into two portions, the portion (14) used for driving the nail, and the "wasted" or, as it is often called, "lost" portion (15). It may, however, happen that the portion (15) does the useful work, while (14) is wasteful. This would be the case, for instance, in molding a rivet with a hammer, or in forging a piece of iron under the blows of a steam-hammer. The useful work here consists in the deformation of the bolt or piece of iron.

It appears from the expressions (14) and (15) that, for the purpose of driving the nail, m should be large in comparison with m', while for molding a rivet it is of advantage to have m' large in comparison with m.

27. In applying the formulæ (11) to (15), and in general all formulæ of theoretical kinetics, it should be noticed that the forces are supposed to be expressed in absolute measure, the unit being the poundal or the dyne. Hence, to find the force F in pounds the numerical result obtained from one of these formulæ must be divided by the value of g.

28. Let us now consider the *change of the total kinetic energy* produced by direct impact in two partially elastic spheres. With the notations of Art. 11, we have for the excess of the kinetic energy after impact over that before impact:

$$\tfrac{1}{2}(mv^2+m'v'^2)-\tfrac{1}{2}(mu^2+m'u'^2).$$

To eliminate v and v' from this expression, square the equations (7) and (10),

$$(mv+m'v')^2=(mu+m'u')^2,$$

$$(v-v')^2=e^2(u-u')^2;$$

multiply the latter by mm', and write it in the form

$$mm'(v-v')^2+(1-e^2)mm'(u-u')^2=mm'(u-u')^2;$$

finally add the former equation,

$$(m+m')(mv^2+m'v'^2)+(1-e^2)mm'(u-u')^2$$
$$=(m+m')(mu^2+m'u'^2),$$

whence

$$\tfrac{1}{2}(mv^2+m'v'^2)-\tfrac{1}{2}(mu^2+m'u'^2)=-\tfrac{1}{2}(1-e^2)\frac{mm'}{m+m'}(u-u')^2. \quad (16)$$

As the right-hand member of this equation is essentially negative, it appears that *while in impact the total momentum remains unchanged, the total kinetic energy is in general diminished;* only in the limiting case of perfectly elastic bodies ($e=1$) does the kinetic energy remain the same after as before impact. The "lost" kinetic energy (16) mainly represents the amount of energy spent in producing the permanent deformation of the impinging bodies.

29. Exercises.

(1) A hammer weighing 1.5 lbs. strikes a nail weighing $\frac{1}{2}$ oz. with a velocity of 20 ft. per second, and drives it $\frac{1}{4}$ in. Find the mean resistance of the wood, and determine the useful and wasteful work.

(2) In Art. 20, Ex. (3), find the loss of kinetic energy due to the impact.

(3) A train of 120 tons runs, with a speed of 15 miles an hour, into a train of 80 tons at rest. Neglecting elasticity, determine the destructive work of the collision, and the velocity along the track after impact.

(4) A pile weighing m' lbs. is driven into the ground by a ram of m lbs., falling from a height of h ft. If the pile sinks s in. into the ground after n falls of the ram, show that the resistance of the ground (assumed as uniform) is $=\frac{12n}{s}\left(\frac{1+e}{1+m'/m}\right)^2 m'h$ pounds.

(5) If, in Ex. (4), the elasticity of ram and pile be neglected, ram and pile will have equal velocities after impact, and move together. Hence, the factor m' should be replaced by $m+m'$, and the resistance is $=\frac{12n}{s}\cdot\frac{1}{1+m'/m}\cdot mh$ pounds.

(6) 10 blows of a ram of 500 lbs., falling from a height of 5 ft., sink a pile of 400 lbs. 4 in. If the permanent load of a pile be taken as one-fifth of the resistance, what permanent load can the pile bear?

(7) A steam-hammer of 3 tons is used in forging. It has a fall of 5 ft. If the weight of the anvil be 20 tons, what is the useful and what the wasteful work?

30. Recoil. The explosion of the powder in a gun produces an impulsive pressure both on the shot and on the body of the gun. Assuming the line of motion of the centroid of the shot to pass through the centroid of the gun, we may apply equation (7), with $u=0$, $u'=0$. Hence, denoting by m the mass of the gun, by m' that of the shot, we find for the velocity of recoil

$$v = -\frac{m'}{m} v'. \tag{17}$$

The kinetic energies $\frac{1}{2} mv^2$ and $\frac{1}{2} m'v'^2$ of gun and shot are in the ratio m'/m; hence, the energy of recoil is the fraction $m'/(m+m')$ of the total energy $\frac{1}{2} mv^2 + \frac{1}{2} m'v'^2$ of the explosion of the powder, while the energy of the shot is $= m/(m+m')$ of the total energy. In large guns the recoil is diminished by a special elastic cushion or "compressor." Moreover, the mass of the powder gases cannot be entirely neglected in all cases.

31. Oblique Impact. In the case of oblique impact, *i.e.* when the centres of the colliding spheres do not move in the same straight line, the velocities after impact can be found without difficulty, provided that the velocities of the centres before impact lie in the same plane and that the spheres are perfectly smooth.

With these assumptions, let m, m' be the masses of the two spheres; C, C' their centres (Fig. 3); u, u' the velocities before impact; α, α' the angles made by u, u' with the line CC'; v, v' the velocities after impact; and β, β' the angles they make with CC'.

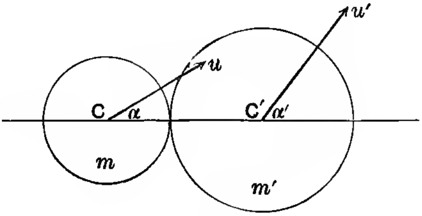

Fig. 3.

As there is no friction, the forces of impact act along the line CC' that joins the centres. Hence, resolving each velocity along and perpendicular to CC', the components at right angles to CC' must remain unchanged by the collision; that is, we must have

$$v \sin\beta = u \sin\alpha, \quad v' \sin\beta' = u' \sin\alpha', \tag{18}$$

The components of the velocities along CC' must satisfy the equations (7) and (10). Hence, substituting $u \cos \alpha$, $u' \cos \alpha'$, $v \cos \beta$, $v' \cos \beta'$ for u, u', v, v', respectively, we must have

$$mv \cos \beta + m'v' \cos \beta' = mu \cos \alpha + m'u' \cos \alpha', \tag{19}$$

$$v' \cos \beta' - v \cos \beta = e(u \cos \alpha - u' \cos \alpha'). \tag{20}$$

32. The particular case of the oblique impact of a homogeneous sphere against a smooth fixed plane deserves special mention. In this case, u' and v' are zero; and the angles α, β made by the velocities u, v with the normal to the plane, are called the *angle of incidence* and *of reflection*, respectively.

The equations (18) and (20) reduce to the following:

$$v \sin \beta = u \sin \alpha, \quad v \cos \beta = -eu \cos \alpha, \tag{21}$$

where the minus sign indicates that the projections of u and v on the normal have opposite sense. Dividing the former of these equations by the latter, we find

$$\tan \alpha = -e \tan \beta, \tag{22}$$

where the minus sign merely indicates that the angles α, β lie on opposite sides of the normal.

For perfectly elastic bodies, the last equation shows that the angles of incidence and reflection are equal.

33. Exercises.

(1) A baseball weighing $5\frac{1}{4}$ oz., while moving with a velocity of 100 ft. per second is struck by the bat in a direction at right angles to its line of motion. Find the momentum imparted by the blow if it deflects the ball through an angle of 60°.

(2) Determine the velocity of recoil of a gun weighing 1500 lbs. when a 12-lb. shot is fired from it with an initial velocity of 2000 ft. per second.

(3) The heavier one of two ivory balls ($e = 0.88$), whose centroids are C, C' and whose masses are 1 lb. and $\frac{3}{4}$ lb., impinges upon the lighter. The velocity of the heavier ball is 15 ft. per second and makes an angle of 30° with the line CC', while the velocity of the lighter ball is 5 ft. per second and makes an angle of 75° with the line CC' (produced). Find the velocities after impact in magnitude and direction.

34. As a more careful study of the theory of impact requires some knowledge of the theory of elasticity, it is generally treated more at length in works on applied mechanics. See, for instance, J. Weisbach, *Mechanics of engineering*, translated by E. B. Coxe, New York, Van Nostrand, 1875, Vol. I., pp. 667–711; A. Ritter, *Technische Mechanik*, Leipzig, Baumgärtner, 1884, pp. 585–618; J. H. Cotterill, *Applied Mechanics*, London, Macmillan, 1884, pp. 274–280 and 374–386; Thomson and Tait, *Natural philosophy*, I., Part 1, pp. 274–284. The general theory of impulsive forces will be given in Chapter VI.

II. *Rectilinear Motion.*

35. The motion of a single particle presents a comparatively simple problem, because the forces, being in this case all applied at one and the same point, have a single resultant which is readily found by geometrically adding the forces (Part II., Art. 96). Let this resultant be denoted by F, the mass of the particle by m, and its acceleration by j; then, according to the definition of force (Part II., Art. 60), we must have

$$mj = F.$$

This equation merely expresses the fact that the force F produces in the mass m an acceleration j, which agrees with F in direction and sense, and is inversely proportional to m.

36. The forces, whose resultant is F, are usually called the **impressed forces.** Both F and j are, in general, variable. If at any time t the particle m were acted upon by a force $= -mj$, in addition to the impressed forces, it would evidently be in equilibrium. The product mj of the mass of the particle into its acceleration at any instant is called the **effective force** of the particle at this instant. It can, therefore, be said that *the impressed forces are at any instant in equilibrium with the effective force reversed.*

This obvious proposition forms the fundamental idea of a most important method of treating the dynamical equations of motion, known as **d'Alembert's principle,** which will be discussed more fully later on (see Arts. 97–103, 383–386). It makes it possible to apply to kinetic problems the statical conditions of equilibrium. Thus, in the case of a single particle, if the reversed effective force, $-mj$, be combined with the impressed forces, we have a system of forces acting on the particle which, at the instant considered, is in equilibrium, and must satisfy the conditions of equilibrium for concurrent forces (Part II., Arts. 97, 101), viz. $-mj + F = 0$; or, resolving j into its components d^2x/dt^2, d^2y/dt^2, d^2z/dt^2, and F into the components X, Y, Z,

$$-m\frac{d^2x}{dt^2}+X=0,\quad -m\frac{d^2y}{dt^2}+Y=0,\quad -m\frac{d^2z}{dt^2}+Z=0.$$

37. To familiarize the student with the idea of force and its use in kinetics, we shall now study in some detail the simple case of rectilinear motion. The next section will be devoted to the general problem of the curvilinear motion of a *free* particle. This will be illustrated by the important case of motion due to central forces. Finally, the motion of a particle subject to conditions, or *constraints*, will be treated.

38. When a particle of mass m moves in a straight line, both its velocity v and the resultant force F must be directed along this line. The acceleration in rectilinear motion (see Part I., Art. 103) is $j=dv/dt=d^2s/dt^2$; hence *the dynamical equation of rectilinear motion*,

$$m\frac{dv}{dt}=F, \text{ or } m\frac{d^2s}{dt^2}=F. \qquad (1)$$

It differs from the kinematical equation (Part I., Art. 115) only by the factor m, and can be treated in the same way.

Thus, if the law of force be given, *i.e.* if F be known as a function of t, s, v, or of only one or two of these quantities, the equation can be integrated; and if, moreover, the initial position and velocity of the particle be given, the constants of integration can be determined, and all the circumstances of the motion can be found.

If the mass m of the moving particle were not a constant quantity, the equation (1) should be written in the form

$$\frac{d(mv)}{dt}=F,$$

since the resultant force is the rate at which the momentum of the particle changes with the time (see Part II., Art. 60).

39. As long as a single free particle only is considered, there is generally no advantage in introducing the idea of force; the equation of motion can be divided by m, and this reduces it to a purely kinematical form.

Thus, for a particle of mass m falling *in vacuo*, the dynamical equation of motion is

$$m\frac{d^2s}{dt^2}=W,$$

where $W=mg$ is the weight of the particle; *i.e.* the force of attraction exerted by the earth on the particle (in poundals or dynes, if m be expressed in pounds or grammes, see Part II., Art. 115). Dividing by m, we find the kinematical equation

$$\frac{d^2s}{dt^2}=g,$$

which has been treated in Part I., Arts. 107–114.

The following articles give examples in which it is more convenient to retain the idea of force.

40. Let us consider a mass m that is being raised or lowered by means of a rope or chain (Fig. 4), such as a building stone suspended from a derrick. The rope acts as a *constraint*, conditioning the motion of the stone. To make the stone free we may imagine the rope cut just above the stone and the **tension** of the rope, T, introduced as a substitute. The stone then moves under the action of two forces, viz. its weight $W=mg$ and the tension T of the rope. Taking the downward sense as positive, we have the equation of motion,

$$m\frac{d^2s}{dt^2}=mg-T. \tag{2}$$

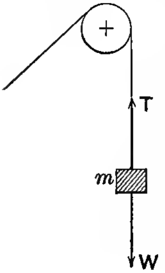

Fig. 4.

41. Writing j for the acceleration d^2s/dt^2 with which the stone is being lowered or raised, we find for the tension T of the rope

$$T=m(g-j). \tag{3}$$

This equation shows that the tension is equal to the weight of the stone, not only when it is hanging at rest, but also whenever it is raised or lowered with constant velocity; and that the tension is zero if the stone is lowered with an acceleration equal to that of gravity, as is otherwise evident.

42. The above formula will give the tension T in poundals (or dynes), if the mass m be expressed in pounds (or grammes), and the accelerations in feet (or centimetres) per second per second.

In engineering practice, gravitation measure is commonly used for weights as well as for the forces that replace constraints (tensions, pressures, friction, etc.). The engineer would, therefore, divide by g the numerical value of T just found, so as to reduce it to pounds.

It should be noticed that the general equations of theoretical mechanics are of course independent of the system of units adopted, and that in applying them to numerical examples it is only necessary to use one and the same system of units consistently throughout. As modern physics has settled upon mass as a fundamental unit (see Part II., Art. 68), regarding the unit of force as derived from and based upon the unit of mass, this absolute system will always be adopted in this book, unless the contrary be specified. In other words, *it will always be assumed that mass is expressed in pounds* (or grammes), *and consequently force in poundals* (or dynes).

43. Let us next consider two particles, m_1, m_2, connected by a cord hung over a vertical fixed pulley, as in the apparatus known as *Atwood's machine* (Fig. 5). If $m_1 > m_2$, m_1 will descend while m_2 ascends. The effective force of the system formed by the two particles is evidently the difference of the weights of the particles, viz. $W_1 - W_2 = (m_1 - m_2)g$, while the whole mass moved (neglecting the mass of the cord and of the pulley) is m_1+m_2. Hence, we have for the acceleration j of the system,

$$j=\frac{m_1-m_2}{m_1+m_2}g. \tag{4}$$

This acceleration is constant, and the relations between space, time, and velocity are found just as for a single particle falling freely, except that the acceleration of gravity g is replaced by the fraction $(m_1-m_2)/(m_1+m_2)$ of g. It follows that if the masses m_1, m_2 be selected nearly equal, the acceleration will be small, and the motion can be observed more conveniently than that of a freely falling body.

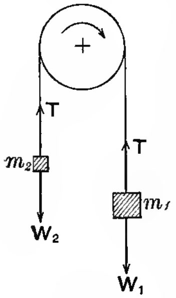

Fig. 5.

44. The tension T of the cord is, of course, the same at every point of the cord if, as is here assumed, the weight of the cord and the axle-friction of the pulley be neglected. To determine this tension, we have only to consider either particle separately.

If the cord be cut just above m_1, and the tension T be introduced, the particle m_1 will move like a free particle under the action of the resultant force $W_1-T=m_1g-T$. Hence, as the sense of the acceleration j of m_1 agrees with that of g,

$$j=\frac{m_1g-T}{m_1}. \tag{5}$$

Similarly, we have for the acceleration of m_2, whose sense is opposite to that of g,

$$j=-\frac{m_2g-T}{m_2}. \tag{6}$$

Eliminating j between any two of the equations (4), (5), (6), we find the tension

$$T=\frac{2\,m_1m_2}{m_1+m_2}g. \tag{7}$$

45. If the two particles of Art. 43 move on inclined planes intersecting in the horizontal axis of the pulley (Fig. 6), it is only necessary to resolve the weights m_1g and m_2g into two components, one parallel, the other perpendicular, to the inclined plane. If the planes be smooth, the system formed by the two particles is made free by introducing the normal reactions of the planes which counterbalance the perpendicular components of the weights. The effective force is therefore the difference of the parallel components, and the acceleration is

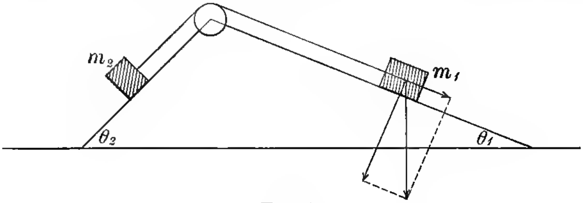

Fig. 6.

$$j=\frac{m_1\sin\theta_1-m_2\sin\theta_2}{m_1+m_2}g, \tag{8}$$

where θ_1, θ_2 are the angles of inclination of the planes to the horizon.

The tension T of the connecting cord is again determined by equating this value of j to the one obtained by considering either of the two particles separately. Thus, m_1 taken by itself, becomes free if we introduce not only the normal reaction of the plane, but also the tension of the string. This gives

$$j=\frac{m_1g\sin\theta_1-T}{m_1}. \tag{9}$$

Similarly, we have for m_2

$$j=\frac{T-m_2g\sin\theta_2}{m_2}. \tag{10}$$

Hence, $$T=\frac{m_1m_2}{m_1+m_2}g(\sin\theta_1+\sin\theta_2). \tag{11}$$

With $\theta_1=\theta_2=\pi/2$ the formulæ (8) to (11) reduce, of course, to the formulæ (4) to (7).

46. Exercises.

(1) A stone weighing 200 lbs. is raised vertically by means of a chain running over a fixed pulley. Determine the tension of the chain: (*a*) when the motion is uniform; (*b*) when the motion is uniformly accelerated upwards at the rate of 8 ft. per second; (*c*) when the acceleration is 32 ft. per second downwards. Neglect the weight of the chain and the axle-friction of the pulley.

(2) A railroad car weighing 4 tons is pushed by four men over a smooth horizontal track. If each man exerts a constant pressure of 100 pounds, (*a*) what is the velocity acquired by the car at the end of 5 sec.? (*b*) what is the distance passed over in these 5 sec.?

(3) (*a*) Determine the constant force required to give a train of 90 tons a velocity of 30 miles an hour in 5 min. after starting from rest. (*b*) How far does the train go in this time? (*c*) If the same velocity is to be acquired at the end of the first mile, what must be the tractive force of the engine?

(4) If in Atwood's machine (Fig. 5) the two masses are each 2 lbs., and an additional mass of 1 oz. be placed on one of these masses, how long will it take this mass to descend 6 ft.? ($g = 32.2$.)

(5) If in Ex. (4) an additional mass of half an ounce be placed on *each* of the 2 lb. masses, how would the tension in the cord differ from the tension in Ex. (4)?

(6) Solve the problem of Art. 45 when the inclined planes are rough, the coefficients of friction being μ_1, μ_2.

(7) A mass of 5 lbs. rests on a smooth horizontal table, and has a cord attached which runs over a smooth pulley on the edge of the table. If a mass of 1 lb. be suspended from the cord, find the acceleration and the tension of the cord.

(8) A sleigh weighing 500 lbs. is drawn over a horizontal road, the coefficient of friction being $\frac{1}{70}$. Find the pull exerted by the horses when the motion is uniform.

(9) When the U.S.S. *Raritan* was launched she was observed to pass in 11 sec. over an incline of $3^\circ 40'$, 54 ft. long. Find the coefficient of friction.

(10) A coaster, after coming down a hill, runs up another hill a distance of 200 ft. (from its foot) in 10 sec., when it stops. If the slope of the second hill be 6°, find the coefficient of friction.

(11) A train of 120 tons is running 25 miles an hour. Find what constant force is required to bring it to rest: (*a*) in 3 min.; (*b*) in half a mile.

(12) If it takes 1 min. to coast down a hill on a uniformly sloping road of 1 mile length, and the coefficient of friction be 0.02, what is the height of the hill?

47. Kinetic Energy and Work. The dynamical equation of motion, (1), Art. 38, can often be integrated after multiplying both members by $vdt = ds$; this makes the left-hand member an exact differential, viz. the differential of the *kinetic energy* $\frac{1}{2}mv^2$, while the right-hand member, Fds, represents the *elementary work* done by the force F:

$$d(\tfrac{1}{2}\,mv^2) = Fds.$$

If F be given as a function of s alone, this equation can be integrated, say from the time t_0 to the time t. Denoting by s_0 and v_0 the values of s and v at the time t_0 (Fig. 7), we find

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \int_{s_0}^{s} Fds. \qquad (12)$$

This equation gives the velocity v as a function of the distance s, counted from the arbitrary origin O. As $v = ds/dt$, a second

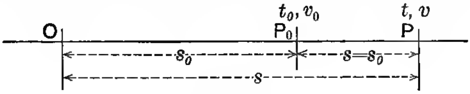

Fig. 7.

integration will give s as a function of t. Examples of this method have been given in Part I., Arts. 109, 117, 119, 121, 122; it will here only be necessary to call attention to the dynamical meaning of the quantities involved.

48. The left-hand member of equation (13) represents evidently the increase in the kinetic energy of the moving particle, while the right-hand member expresses the work done by the force F during the passage of the particle from the point P_0 to the point P (see Part II., Arts. 71, 72). Hence, the meaning of the equation is that *the increase in the kinetic energy is equal to the work done by the resultant force.* This is the **principal of work** or **of kinetic energy** (or of *vis viva*) for the case of the rectilinear motion of a particle.

Thus, for a falling body, F is constant and equal to the weight mg of the body; hence, equation (12) gives, if s be counted positive downwards,

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=mg(s-s_0),$$

where the right-hand member represents the work done by the weight of the body, *i.e.* by the attractive force of the earth during the fall of the body through the distance $s-s_0$.

For a body thrown vertically upwards with an initial velocity v_0, we have

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=-mgs,$$

if s be counted from the starting point and positive upwards. The kinetic energy here decreases, the initial kinetic energy, $\tfrac{1}{2}mv_0^2$, being, so to speak, consumed by the work done against the force of gravity.

49. Inclined Plane. When a particle of mass m is moved uniformly up a smooth inclined plane from P_0 to P_1 (Fig. 8), the work done against gravity is equal to the work that would have to be done in raising the particle m through the vertical height PP_1 of P_1 above the initial point P_0. For, putting $P_0P_1=s$, $PP_1=h$, and denoting the inclination of the plane to the horizon by θ, we have for the work,

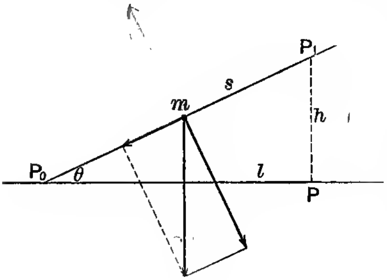

Fig. 8.

$$mg\sin\theta\cdot s=mg\cdot s\sin\theta=mg\cdot h.$$

If the plane be rough, the coefficient of friction being μ, the effective force for motion upwards is $=mg\sin\theta+\mu mg\cos\theta$; hence, the work done in moving the mass m from P_0 to P_1 is

$$mgs\sin\theta+\mu mgs\cos\theta=mg\cdot h+\mu mg\cdot l=mg(h+\mu l),$$

where $l=P_0P$ is the horizontal distance of the final position P_1 from the starting point P_0. The total work is, therefore, the sum of the work of overcoming gravity through the *vertical* distance h and the work of overcoming friction through the *horizontal* distance l.

50. Work done on a System of Particles. Let there be given any number of particles of masses $m_1, m_2, \dots m_n$ at the distances $s_1, s_2, \dots s_n$ above a fixed horizontal plane; and let these masses be raised vertically against gravity so that their distances from the same plane become $s_1', s_2', \dots s_n'$. The centroid of the masses in their original position has a distance $\bar{s}=\Sigma ms/\Sigma m$ from the fixed plane, while in the final position it has the distance $\bar{s}'=\Sigma ms'/\Sigma m$ from the same plane. It has, therefore, been raised through a distance $\bar{s}'-\bar{s}$. It follows that *the total work done in raising the separate masses*, viz.

$$m_1g(s_1'-s_1)+m_2g(s_2'-s_2)+\cdots+m_ng(s_n'-s_n)=g(\Sigma ms'-\Sigma ms),$$

is equal to the work that would be done in raising the total mass Σm *through the distance* $\bar{s}'-\bar{s}$ *traversed by the centroid, i.e.* to

$$g\Sigma m\cdot(\bar{s}'-\bar{s}).$$

51. The Work of a Variable Force is well illustrated by the expansion of gas or steam in a cylinder with a movable piston (Fig. 9). Let r be the radius of the cylinder, p the pressure (in pounds) at any instant of the gas per square inch of surface; then the total pressure of the gas on the inside of the piston is $P=\pi r^2p$ pounds, and if P_0 be the pressure on the outside (say the atmospheric pressure), the effective force acting on the piston is $F=P-P_0$, friction being neglected.

The force F is variable, since the pressure p varies with the volume v occupied by the gas. This volume being in the present case proportional to the distance s of the piston from the fixed base of the cylinder, the force F is a function of s. The variation of F can therefore be represented graphically by a curve having s for abscissa and F for ordinate (Fig. 9); and *the area of this curve, i.e.* the area contained between the curve, the axis of s, and two ordinates whose abscissas are s_0 and s, being given by the integral $\int_{s_0}^{s} F ds$, *represents the work done on the piston when pushed through the distance* $s - s_0$.

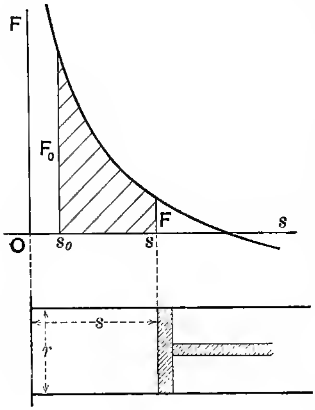

Fig. 9.

52. In the case of a perfect gas, Boyle's law gives the relation $pv = k$, where k is constant if the temperature remains constant. Hence,

$$F = \frac{k}{s} - P_0,$$

where k and P_0 are constants. This equation represents an equilateral hyperbola, whose asymptotes are the axis of F and a line parallel to the axis of s. For steam, the law connecting pressure and volume is more complicated, but the curve may be taken as very nearly hyperbolic.

53. The Steam-engine Indicator is an apparatus for measuring the pressure of the steam in the cylinder and at the same time recording it automatically on a drum revolving as the piston moves. Thus, if the indicator be put in connection with the interior of the cylinder, the curve traced by the indicator has for its abscissas the distances s of the piston from this end, and for its ordinates the corresponding pressures F of the steam on the inside of the piston.

At the beginning of the stroke, steam is admitted and acts with nearly constant pressure on the piston; the line AB (Fig. 10) traced by the indicator will therefore be nearly parallel to the axis of s. As soon as the steam is shut off by the slide-valve, the steam, being now confined within the cylinder, begins to expand nearly according to the law $pv =$ const., or $Fs =$ const.; the curve traced by the indicator is therefore approximately an equilateral hyperbola BC, having the axes as asymptotes. When the slide-valve connects the cylinder with the condenser, a partial vacuum is established behind the piston, and the pressure curve is approximately a line CD, parallel to the axis of F.

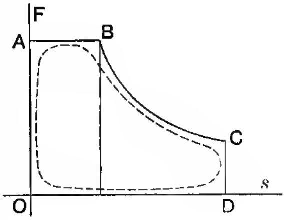

Fig. 10.

54. The area $ABCDO$ evidently represents approximately the work of the pressure on the inside of the piston in one complete (forward and backward) stroke. In reality, a large number of circumstances produce deviations from the regular shape $ABCDO$, and the actual trace, obtained by means of an indicator for one (forward and backward) stroke, usually called the *indicator diagram*, forms a loop somewhat like that indicated by the dotted curve in Fig. 10. The area of this loop, which represents the work in question, can readily be found by dividing it up into narrow rectangular strips, or with the aid of a planimeter.

55. The *effective* piston pressure is of course the difference between the pressures on the two sides of the piston. A diagram should therefore be obtained for each side of the piston; from these two diagrams the curve of effective piston pressure is then derived by constructing the curve whose ordinates are the differences of the corresponding pressures on the two sides. By dividing the area contained between this curve and the axes by the length of the stroke, the average, or *mean, piston pressure* is finally found.

For details the student is referred to special works on the steam engine, such as G. C. V. Holmes, *The steam engine,* New York, Appleton, 1887, pp. 317–345.

56. Attractive and Repulsive Forces. Let us consider the motion of a particle acted upon by a so-called *central force, i.e.*

a force whose direction constantly passes through a fixed centre *O*, while its magnitude is a function of the distance s from the centre alone. If the initial velocity be zero, or if its direction pass through the centre *O*, the motion of the particle will be rectilinear, the line of motion passing through the centre of force, *O*. The most important special cases of this kind have been treated in kinematics (Part I., Arts. 117–124, 176).

57. Let the force be due to a mass m' concentrated at the centre *O*, and attracting according to *Newton's law of the inverse square of the distance* (Part II., Art. 257). Counting the distances s from the centre *O* as origin (Fig. 11), we have, for the force acting on the particle m,

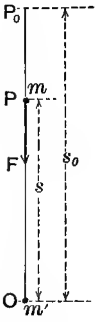

Fig. 11.

$$F = -\kappa \frac{mm'}{s^2},$$

where κ is a constant whose value may be determined, as indicated in Part I., Art. 119, and Part II., Arts. 262, 263.

The principle of kinetic energy, equation (12), Art. 47, gives at once

$$\tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_0{}^2 = -\kappa m m' \int_{s_0}^{s} \frac{ds}{s^2} = \kappa m \left(\frac{m'}{s} - \frac{m'}{s_0} \right).$$

The quantity m'/s, or in absolute measure $\kappa m'/s$, is the potential at *P* due to m' (Part II., Art. 278), and $\kappa m'/s_0$ is the potential at P_0 due to the same mass m'. The increase in kinetic energy is, therefore, proportional to the decrease in potential.

The quantity $\kappa mm'/s$ is sometimes called the *mutual potential* of the masses m and m'; hence, the increase of the kinetic energy can be said to be equal to the difference of the mutual potentials in the final and initial positions.

The negative of the mutual potential is designated as the

potential energy of the moving particle m. Denoting this by V, and the kinetic energy by T, the last equation becomes

$$T-T_0=-V+V_0,$$

or

$$T+V=T_0+V_0=\text{const.};$$

i.e. the sum of the kinetic and potential energies remains constant during the motion. This is **the principle of the conservation of energy** for this particular problem.

58. It is easy to see that the principle of the conservation of energy holds generally whenever the resulting force F is a function of the distance s alone.

Indeed, if $F=F(s)$, the principle of kinetic energy gives

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=\int_{s_0}^{s}F(s)ds; \qquad (13)$$

hence, putting $\int F(s)ds=f(s)$, where $f(s)$ is called the *force-function*, or potential function, while $-f(s)$ is the *potential energy*, we have

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=f(s)-f(s_0), \qquad (14)$$

or, with the notation of Art. 57,

$$T+V=T_0+V_0=\text{const.} \qquad (15)$$

59. When the resultant force F is an *attraction directly proportional to the distance* s from a fixed centre O, say

$$F=-m\kappa^2s,$$

the potential energy is, by Art. 58,

$$V=-f(s)=\tfrac{1}{2}m\kappa^2s^2.$$

Hence, the principle of the conservation of energy gives

$$v^2+\kappa^2s^2=\text{const.};$$

or, if the initial velocity is zero when $s=s_0$,

$$v=-\kappa\sqrt{s_0^2-s^2}.$$

60. Tension of an Elastic String. According to Hooke's law, the tension of an elastic string is, within the limits of elasticity (*i.e.* as long as no *permanent* deformation is produced), directly proportional to the extension or change of length produced.

Thus, let an elastic string, whose natural length is l, assume the length s when its tension is T; then Hooke's law can be expressed in the form

$$T = k(s - l),$$

where k is a constant. To determine this constant for a given string, we may observe the length l_1 assumed by the string under a known tension, say the tension $T_1 = mg$, produced by suspending a given mass m from the string (the weight of the string itself being neglected). We then have

$$T_1 = k(l_1 - l).$$

Hence, dividing

$$\frac{T}{mg} = \frac{s - l}{l_1 - l},$$

or, denoting by e the extension $l_1 - l$ due to the weight mg,

$$T = \frac{mg}{e}(s - l). \qquad (16)$$

61. By means of this relation we can determine the motion of a particle of mass m attached to a fixed point O by means of an elastic string, if the string be stretched and then let go. We shall assume the particle and string to lie on a smooth horizontal table, so as to eliminate the effect of the weight of the particle.

The equation of motion is

$$m\frac{d^2s}{dt^2} = -\frac{mg}{e}(s - l), \qquad (17)$$

whence, putting for shortness $\sqrt{g/e} = \kappa$,

$$s = l + C_1 \cos \kappa t + C_2 \sin \kappa t,$$

$$v = \frac{ds}{dt} = -\kappa C_1 \sin \kappa t + \kappa C_2 \cos \kappa t.$$

If the initial length of the string at the time $t = 0$ be s_0, the constants are readily determined, and we find

$$s = l + (s_0 - l) \cos \kappa t,$$

$$v = -\kappa (s_0 - l) \sin \kappa t. \qquad (18)$$

It should be noticed that these equations hold only as long as the string is actually stretched, *i.e.* as long as $s > l$; the motion that ensues when s becomes less than l is, however, easily determined from the velocity for $s = l$.

62. Exercises.

(1) In a steam engine, let $p = 15$ lbs. per square inch be the mean piston pressure during one stroke, $s = 4$ ft. the length of the stroke, and $d = 1.5$ ft. the diameter of the cylinder. (*a*) What is the work per stroke? (*b*) To what height could a mass of 500 lbs. be raised by this work?

(2) A train of 80 tons starting from rest acquires a velocity of 30 miles an hour on a level road at the end of the first mile. Determine the average tractive force of the engine: (*a*) if the frictional resistances be neglected; (*b*) if these resistances be estimated at 8 lbs. per ton. (*c*) What tractive force is required to haul the same train over a level road at a constant speed?

(3) A train of 60 tons runs one mile with constant speed; if the resistances be 8 lbs. per ton, find the work done by the engine: (*a*) on a level track; (*b*) on an average grade of 1%. (*c*) On a 1% grade, what is the ratio of the work done against gravity to that done against the resistances?

(4) Determine the work expended in raising from the ground the materials for a brick wall 30 ft. high, 40 ft. long, and 2 ft. thick, the weight of a cubic foot of brickwork being 112 lbs.

(5) Knowing that on the surface of the earth the attraction per unit of mass is $g = 32$, find what it would be on the sun if the density of the sun be $\frac{1}{4}$ of that of the earth, and its diameter 108 times that of the earth.

(6) Show that the velocity acquired by a body in falling to the surface of the earth from an infinite distance, under the action of the earth's attraction alone, would be $v = \sqrt{2gR}$, or about 7 miles per second (with $R = 4000$ miles).

(7) A homogeneous straight rod, $AB = l$, of constant density ρ, attracts a particle P of mass 1 according to the law of the inverse square of the distance. The initial position P_0 of P is on AB produced beyond B, at the distance $BP_0 = s_0$, and the initial velocity is zero.

(*a*) Determine the velocity v of P at any distance $BP = s$, and its velocity v_1 at B. (*b*) How is the solution to be modified if the linear mass BA extends from B to infinity?

(8) A circular wire of radius a and constant density ρ attracts, according to Newton's law, a particle P of mass 1, situated on the *axis* of the circle; *i.e.* on the perpendicular to its plane passing through the centre O. If the velocity is zero when the particle is at the distance $OP_0 = s_0$, determine the velocity of the particle at any distance s, and show that the motion is oscillatory.

(9) Determine the motion of two free particles of masses m_1, m_2, attracting each other according to Newton's law, and starting at the distance s_0 with zero velocity.

(10) Show that the motion of the particle in Art. 61 is oscillatory, and that the period, *i.e.* the time of one complete oscillation, is

$$= 2\sqrt{e/g}[\pi + 2\,l/(s_0 - l)].$$

(11) A particle of mass m is suspended from a fixed point by means of an elastic string whose weight is neglected. The natural length of the string is l. Its length, when the mass m is suspended at its end, is l_1. If the particle be pulled down so as to make the length of the string $= s_0$, and then released, the particle will perform oscillations. Determine their period: (*a*) if $s_0 - l_1 < l_1 - l$; (*b*) if $s_0 - l_1 > l_1 - l$.

(12) The particle in Ex. (11) is raised through a height h, so as to loosen the string, and then dropped. Determine the greatest extension of the string.

(13) An elastic string, whose natural length is $= l$, is suspended from a fixed point. A mass m_1 attached to its lower end stretches it to a length l_1; another mass m_2 stretches it to a length l_2. If both these masses be attached and then the mass m_2 be cut off, what will be the motion of m_1?

(14) A particle performs rectilinear oscillations owing to a centre of force in the line of motion attracting the particle with a force directly proportional to the distance. The motion of the particle is impeded by a resistance directly proportional to the velocity. Investigate the motion.

63. Power. It has been shown that the time-effect of a force is measured by its *impulse* (Art. 2), while the space-effect is measured by its *work* (Arts. 47, 48). In applied mechanics it is of great importance to take time and space into account simultaneously. *The time-rate at which work is performed by a force* has therefore received a special name, **power.** The source from which the force for doing useful work is derived is commonly called the *agent;* and it is customary to speak of the power of an agent, this meaning the rate at which the agent is capable of supplying useful work.

64. The *dimensions* of power are evidently ML^2T^{-3}. The *unit of power* is the power of an agent that does unit work in unit time. Hence, in absolute measure, it is the power of an agent doing one erg per second in the C.G.S. system, and one foot-poundal per second in the F.P.S. system. As, however, the idea of power is of importance mainly in engineering practice, power is usually measured in gravitation units. In this case, the unit of power is the power of an agent doing one foot-pound per second in the F.P.S. system, and one kilogramme-metre in the metric system.

A larger unit is frequently found more convenient. For this reason, the name **horse-power** (H.P.) is given to the power of doing 550 foot-pounds of work per second, or $550 \times 60 = 33{,}000$ foot-pounds per minute.

65. Efficiency of Machines. While the principle of the conservation of energy was proved in Arts. 57 and 58 only for a special case, it is known to be of almost universal application to the forces occurring in nature. Thus, in particular in the case of machines it is found to be verified with a degree of approximation corresponding to the precision of the investigation.

The principle can here be expressed in the form

$$W = W_0 + W_1,$$

if W denote the *total work* done by the agent driving the machine (such as animal force, the expansive force of steam, the pressure of the wind, etc.); W_0 the so-called *lost* or *wasteful work* spent in overcoming friction and other passive resistances of the machine; and W_1 the *useful* work done by the machine.

While W and W_1 allow of precise determination, it is in general difficult to determine W_0 accurately; but it is found that the more exactly in any given machine W_0 is determined, the more nearly will the equation $W = W_0 + W_1$ be fulfilled.

As explained in Part II., Art. 255, the ratio W_1/W of the useful work to the total work is called the *efficiency* of the machine. The term *modulus* is used sometimes for efficiency.

66. Exercises.

(1) In electrical engineering a *watt* is defined as the power of doing one joule, *i.e.* 10^7 ergs, per second. Find the relation between the watt and the horse-power.

(2) In countries using the metric system of weights and measures the horse-power is defined as 75 kilogramme-metres per second. Find its relation to the watt and to the British horse-power.

(3) Find the horse-power of the engine in Art. 62, Ex. (1), if it make 1 stroke per second.

(4) The cylinder of a steam engine has a diameter of 15 in.; the stroke is 3 ft.; the number of strokes per minute is 77; the mean pressure of the steam is 40 lbs. per square inch. What is the horse-power of the engine?

(5) Find the horse-power required of the locomotive to haul a train of 100 tons at the rate of 30 miles an hour, the resistances amounting to 8 lbs. per ton: (*a*) on a level road; (*b*) up a 1% grade; (*c*) up a 2% grade.

(6) How much water can an engine furnishing 50 H.P. raise per minute from the bottom of a mine 1000 ft. deep?

(7) The diameter of the cylinder of a steam engine is 30 in.; the stroke 4 ft.; the mean pressure 15 lbs. per square inch; the number of revolutions 24 per minute. If the efficiency of the engine be $\frac{3}{5}$, what is the amount of water raised per hour from a depth of 250 ft.?

(8) In what time would an engine yielding 2 H.P. perform the work of raising the brickwork in Art. 62, Ex. (4)?

(9) A shaft of 8 ft. diameter is to be sunk to a depth of 420 ft. through a material whose specific gravity is 2·2. Determine: (*a*) the total work of raising the material to the surface; (*b*) the time in which it can be done by an engine yielding 3·5 H.P.; (*c*) the time in which it can be done by 4 men working in a capstan, if each laborer does 2500 ft.-lbs. per minute, working 8 hours per day.

III. *Free Curvilinear Motion.*

I. GENERAL PRINCIPLES.

67. Let j be the acceleration of a particle of mass m at the time t; F the resultant of all the forces acting on the particle; then its equation of motion is (Art. 35)

$$mj = F.$$

In curvilinear motion (Fig. 12) the direction of j and F differs from the direction of the velocity v; and the angle ψ between j and v varies in general in the course of time. As shown in kinematics (Part I., Art. 159), the acceleration can be resolved into a tangential component $j_t = dv/dt = d^2s/dt^2$ and a normal component $j_n = v^2/\rho$, where ρ is the radius of curvature of the path. Hence, if the resultant force F which has the direction of j be resolved into a *tangential force* $F_t = F\cos\psi$, and a *normal force* $F_n = F\sin\psi$, the above equation of motion will be replaced by the following two equations:

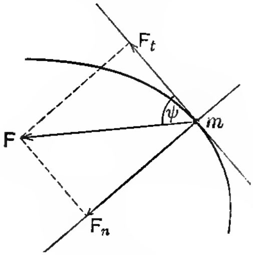

Fig. 12.

$$m\frac{dv}{dt} = F_t, \quad m\frac{v^2}{\rho} = F_n. \tag{1}$$

In the particular case when the normal component F_n is constantly directed towards a fixed point it is called *centripetal force.*

68. The formulæ (1) show how the force F affects the velocity of the particle and the curvature of the path. The change of the *magnitude* of the velocity is due to the tangential force F_t alone. If this component be zero, *i.e.* if the resultant force F be constantly normal to the path, the velocity v will remain of constant magnitude. The curvature of the path, $1/\rho$, and hence the *direction* of v, depends on the normal component F_n. If this component be zero, the curvature is zero; *i.e.* the path is rectilinear.

69. Instead of resolving the resultant force F along the tangent and normal, it is often more convenient to resolve it into three components, $F\cos\alpha = X$, $F\cos\beta = Y$, $F\cos\gamma = Z$, parallel to three fixed rectangular axes of co-ordinates Ox, Oy, Oz, to which the whole motion is then referred. If x, y, z be the co-ordinates of the particle m at the time t, the equations of motion assume the form

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z. \tag{2}$$

Thus, the curvilinear motion is replaced by three rectilinear motions.

70. If the components X, Y, Z were given as functions of the time t alone, each of the three equations (2) could be integrated separately. In general, however, these components will be functions of the co-ordinates, and perhaps also of the velocity and time. No general rules can be given for integrating the equations in this case. By combining the equations (2) in such a way as to produce exact derivatives in the resulting equation, it is sometimes possible to effect an integration. Two methods of this kind have been indicated for the case of two dimensions in a particular example in Part I., Art. 232. We now proceed to study these *principles* of integration from a more general point of view, and to point out the physical meaning of the expressions involved.

71. The Principle of Kinetic Energy. Let us combine the equations of motion (2) by multiplying them by dx/dt, dy/dt, dz/dt respectively, and then adding. The left-hand member of the resulting equation will be the derivative with respect to t of

$$\tfrac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = \tfrac{1}{2}mv^2.$$

We find, therefore,

$$\frac{d(\frac{1}{2}mv^2)}{dt}=X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt},$$

or, multiplying by dt and integrating,

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=\int(Xdx+Ydy+Zdz), \tag{3}$$

where v_0 is the initial velocity.

The left-hand member represents the increase in the kinetic energy of the particle; the right-hand member represents the work done by the resultant force; and equation (3) expresses the equality between the work done and the change in the kinetic energy, that is, the *principle of work* or *of kinetic energy* for the curvilinear motion of a particle (comp. Art. 47). Sometimes the name *principle of vis viva* is given to this proposition, the term vis viva, or living force, meaning the same as kinetic energy, or, in older works, twice the kinetic energy.

72. The principle of work can be deduced still more directly from the equations (1). Multiplying the former of these equations by $vdt=ds$, we find

$$d(\tfrac{1}{2}mv^2)=Fds\cos\psi;$$

hence, integrating,

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=\int_{s_0}^{s}Fds\cos\psi, \tag{4}$$

where v_0 is the velocity of the particle at the place specified by s_0 (comp. Part II., Art. 72).

73. The principle of kinetic energy gives a first integral of the equations of motion whenever the integration indicated in the right-hand member of (3) or (4) can be performed. We proceed to investigate under what conditions this integration becomes possible.

In the most general case the components X, Y, Z, in (3), as well as the tangential force $F\cos\psi$ in (4), are functions of the co-ordinates x, y, z, of the velocity, *i.e.* of the time-derivatives of x, y, z, and of the time t. If the motion of the particle were completely known, that is, if we knew its position at every instant, the co-ordinates would be known functions of the time, say

$$x=f_1(t),\ y=f_2(t),\ z=f_3(t).$$

By differentiation the velocities $v_x=dx/dt$, $v_y=dy/dt$, $v_z=dz/dt$ could be found; and, substituting in (3), the integral would assume the form $\int_{t_0}^{t}\phi(t)dt$, so that the work could be determined by evaluating this integral. As, however, the motion of the particle is generally not known beforehand, this motion being just the thing to be determined, the integral cannot be evaluated in the most general case.

74. *If the forces acting on the particle depend only on the position of the particle*, *i.e.* if X, Y, Z are functions of x, y, z alone, *the integral* $\int_{s_0}^{s}(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)$ *can be determined whenever the path of the particle is given.* For the equations of the path, say

$$f_1(x, y, z)=0,\ f_2(x, y, z)=0,$$

make it possible to eliminate two of the three variables x, y, z from under the integral sign, or to express all three in terms of a fourth variable. In either case the function under the integral sign becomes a function of a single variable, and the work of the forces can be found.

75. *If the forces are such as to make the expression* $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ *an exact differential*, say $d\mathrm{U}$, *the integration can evidently be performed without any knowledge of the path of the particle between its initial and final positions.* In this case equation (3) becomes

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=U-U_0, \tag{5}$$

U_0 being the value of U at the initial position, where $v=v_0$. As most of the forces occurring in nature are of this character, this particular case is of great importance, and deserves careful study.

76. The expression $Xdx+Ydy+Zdz$ will be an exact differential whenever there exists a function U of the co-ordinates x, y, z alone (*i.e.* not involving the time or the velocities explicitly), such that

$$\frac{\partial U}{\partial x}=X, \quad \frac{\partial U}{\partial y}=Y, \quad \frac{\partial U}{\partial z}=Z. \tag{6}$$

If these conditions are fulfilled, we have evidently

$$Xdx+Ydy+Zdz=dU.$$

The function U is called the **force-function**, and forces for which a force-function exists are called *conservative forces.*

Hence, *if the forces acting on a particle are conservative,* in other words, *if they have a force-function, the principle of work gives a first integral of the equations of motion.*

77. The conditions (6) for the existence of a force-function U can be put into a different analytical form which is frequently useful. Differentiating the second of the equations (6) with respect to z, the third with respect to y, we find

$$\frac{\partial^2 U}{\partial y \partial z}=\frac{\partial Y}{\partial z}, \quad \frac{\partial^2 U}{\partial z \partial y}=\frac{\partial Z}{\partial y},$$

whence $\partial Y/\partial z=\partial Z/\partial y$. If we proceed in a similar way with the other equations (6), it appears that they can be replaced by the following conditions:

$$\frac{\partial Y}{\partial z}=\frac{\partial Z}{\partial y}, \quad \frac{\partial Z}{\partial x}=\frac{\partial X}{\partial z}, \quad \frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x}. \tag{7}$$

It is shown in works on the differential calculus and differential equations that these equations (7), or the equations (6), which are equivalent to them, are not only the sufficient, but also the necessary, conditions that must be fulfilled to make $Xdx+Ydy+Zdz$ an exact differential.

78. The dynamical meaning of the existence of a force-function U lies mainly in the fact that, if a force-function exists, the work done by the forces as the particle passes from its initial to its final position depends only on these positions, and not on the intervening path. This is at once apparent from equation (5), in which $U - U_0$ represents this work.

It follows that the work of conservative forces is zero if the particle returns finally to its original position, that is, if it describes a closed path, provided that the force-function U is single-valued, an assumption which will here always be made.

79. In the case of central forces inversely proportional to the square of the distance, for which a force-function can always be shown to exist (see Part II., Arts. 278–281), the force-function is usually called the *potential.* The negative of the force-function, say

$$V = -U,$$

is called the **potential energy**. If this quantity be introduced, and the kinetic energy be denoted by T (as in Art. 57), the equation (5) assumes the form

$$T + V = T_0 + V_0, \tag{8}$$

which expresses the **principle of the conservation of energy** for a particle: *the total energy*, i.e. *the sum of the kinetic and potential energies, remains constant throughout the motion whenever there exists a force-function.* In other words, whatever is gained in kinetic is lost in potential energy, and *vice versa.*

80. The name *force-function* is due to Sir William Rowan Hamilton. Some authors use it for $V = -U$, and not for U. With regard to the term *potential*, the usage is still less settled. Some writers use it for U, others for $-U$, nor is its use always restricted to Newtonian forces. Green was the first to give the name *potential function* to the function U; Gauss brought the expression *potential* into common use. Clausius uses "potential function" for what is called above "potential," reserving the latter name for the potential of a system on another system, or on itself. He also uses the term *ergal* for what is called above "potential energy." Several writers have followed him in this terminology.

81. As the force-function U is a function of the co-ordinates x, y, z alone, an equation of the form

$$U=c, \qquad (9)$$

where c is a constant, represents a surface which is the locus of all points of space at which the force-function has the same value c. By giving to c different values, a system of surfaces is obtained, and these surfaces are called **level,** or **equipotential, surfaces.**

82. The values of the derivatives of U at any point $P(x, y, z)$ are proportional to the direction-cosines of the normal to the equipotential surface (9) at P. But, by (6), they are also proportional to the direction-cosines of the resultant force F at this point. It follows that *the resultant force* F *at any point* P *is always normal to the equipotential surface passing through* P.

If the equation of the equipotential surfaces be given, the resultant force F at any point (x, y, z) is readily found, both in magnitude and direction, from its components (6):

$$F^2=X^2+Y^2+Z^2=\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2. \qquad (10)$$

83. As the particle moves in its path from any point P to an infinitely near point P' (Fig. 13), it passes from one equipotential surface $U=c$ to another $U=c'$. Its velocity meets these surfaces at a varying angle, while its acceleration, which has the direction of the resultant force F, is always normal to these surfaces. The work done by F as the particle moves from P to P' is

$$Fds\cos(F, ds)=Fdn,$$

where $PP'=ds$ and $PP''=dn$, P'' being the intersection of the normal at P with the equipotential surface passing through P'. Hence, by Art. 72,

$$d(\tfrac{1}{2}mv^2)=F\cdot dn=dU. \qquad (11)$$

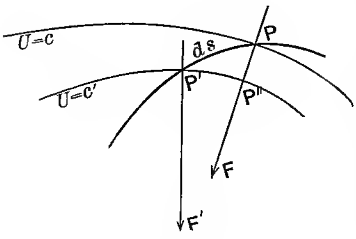

Fig. 13.

The normal distance $PP''=dn$ between two equipotential surfaces is therefore inversely proportional to the force F.

It also appears that whenever the particle in its path returns to the same equipotential surface, the work done by F is zero, and hence, by (5), the velocity assumes the initial value v_0.

84. Let α, β, γ be the direction cosines of F at any point P; λ, μ, ν those of any straight line s drawn through P; and let ϕ be the angle between F and s, so that $\cos\phi=\alpha\lambda+\beta\mu+\gamma\nu$. Then the projection of F on s is

$$F_s=F\cos\phi=F(\alpha\lambda+\beta\mu+\gamma\nu),$$

or, since by (6) $\alpha F=\partial U/\partial x$, $\beta F=\partial U/\partial y$, $\gamma F=\partial U/\partial z$,

$$F_s=\frac{\partial U}{\partial x}\frac{dx}{ds}+\frac{\partial U}{\partial y}\frac{dy}{ds}+\frac{\partial U}{\partial z}\frac{dz}{ds}=\frac{dU}{ds}; \qquad (12)$$

i.e. *the projection of the resultant force on any direction is the derivative of the force-function with respect to that direction.*

This follows also from the equations (6), since the directions of the axes are arbitrary.

If s be taken tangent to the equipotential surface passing through P, we have $F_s=dU/ds=0$; if it be taken normal to this surface, we find $F_s=F=dU/dn$, which agrees with (11).

85. The force-function U determines, as has been shown, a system of equipotential surfaces $U=$const. Starting from a point P on one of these surfaces, say $U=c$ (Fig. 14), let us draw through P the direction of the resultant force, which is normal to the surface $U=c$ (Art. 82). Let this direction intersect in P' the next surface, $U=c'$. At P' draw the normal to $U=c'$, and let it intersect the next surface, $U=c''$, in P''. Proceeding in this way, we obtain a series of points P, P', P'', P''', ..., which in the limit will form a continuous curve whose

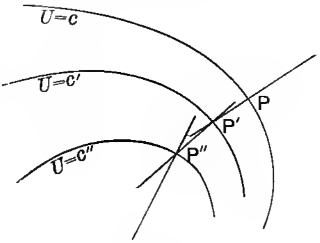

Fig. 14.

direction at any point coincides with the direction of the resultant force at that point. Such a line is called a **line of force.**

The lines of force evidently form the orthogonal system to the system of equipotential surfaces. The differential equations of the lines of force are therefore:

$$\frac{dx}{\frac{\partial U}{\partial x}} = \frac{dy}{\frac{\partial U}{\partial y}} = \frac{dz}{\frac{\partial U}{\partial z}}. \tag{13}$$

86. Exercises.

(1) Show that a force-function exists when the resultant force is constant in magnitude and direction.

(2) Find the force-function in the case of a free particle moving under the action of the constant force of gravity alone (projectile *in vacuo*); determine the equipotential surfaces and the potential energy.

(3) Show the existence of a force-function when the direction of the resultant force is constantly perpendicular to a fixed plane, say the xy-plane, and its magnitude is a given function $f(z)$ of the distance z from the plane.

(4) Find the force-function, the equipotential surfaces, and the kinetic energy when the force is a function $f(r)$ of the perpendicular distance r from a fixed line, and is directed towards this line at right angles to it.

(5) Show that a force-function always exists for a *central force, i.e.* a force passing through a fixed point and depending only on the distance from this point.

(6) Show the existence of a force-function when a particle moves under the action of any number of central forces.

(7) A homogeneous sphere of mass m' attracts a free particle P of mass m with a force $F = \kappa mm'/r^2$, where κ is a constant, and $r = OP$ is the distance of P from the centre of the sphere. Show that the potential is $V = -\kappa mm'/r$, and that the equipotential surfaces are spheres whose common centre is at O.

(8) In Ex. (7), assume $\kappa mm' = 1$, and draw the intersections of the equipotential surfaces with a plane passing through O, from $r = 1$ centimetre to $r = 2$ centimetres, with a difference of potential $= \frac{1}{10}$.

(9) Two spheres, whose masses are as 1 to 2, attract a particle of mass 1 according to Newton's law; the distance of the centres of the spheres is $=4$. Construct the equipotential lines in a plane passing through the centres, by first constructing the equipotential lines for each sphere separately, and then joining the points of intersection whose potential is the same.

(10) A particle of mass m is subject to the force of gravity and to the actions of two fixed centres C_1, C_2, one attracting with a force inversely proportional to the square of the distance, the other repelling with a force directly proportional to the distance. Find the equipotential surfaces.

87. The Principle of Angular Momentum or of Areas. Let us begin with the case of plane motion, the equations of motion being

$$m\frac{d^2x}{dt^2}=X, \quad m\frac{d^2y}{dt^2}=Y.$$

If we combine these equations by multiplying the former by y, the latter by x, and subtracting the former from the latter, we find

$$mx\frac{d^2y}{dt^2}-my\frac{d^2x}{dt^2}=xY-yX. \tag{14}$$

The right-hand member is the moment (with respect to the origin) of the resultant force F whose components are X, Y (see Part II., Art. 91), while the left-hand member is an exact derivative, viz. the derivative with respect to the time of

$$mxdy/dt-mydx/dt,$$

as is easily verified by differentiating this quantity. The result can therefore be written in the form

$$\frac{d}{dt}\left(mx\frac{dy}{dt}-my\frac{dx}{dt}\right)=xY-yX, \tag{15}$$

and gives, if multiplied by dt and integrated,

$$mx\frac{dy}{dt}-my\frac{dx}{dt}=\int(xY-yX)dt. \tag{16}$$

These equations express the principle of angular momentum, or of areas, for plane motion.

88. The name *principle of areas* is due to the kinematical meaning of the left-hand member (comp. Part I., Arts. 227–232). As $xdy-ydx$ represents twice the infinitesimal sector described by the radius vector of the point (x, y) during the element of time, the quantity $xdy/dt-ydx/dt$ is twice the sectorial velocity about the origin. Introducing polar co-ordinates by putting $x=r\cos\theta$, $y=r\sin\theta$, we have $xdy-ydx=r^2d\theta$, and denoting by S the sector described in the time t,

$$2\frac{dS}{dt}=x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}.$$

The kinematical meaning of equation (15), after dividing it by m, can therefore be stated as follows: *the time-rate of change of twice the sectorial velocity about any point is equal to the moment of the acceleration about the same point.*

89. The dynamical meaning of equation (15) appears by considering that mdx/dt, mdy/dt are the components of the momentum mv of the moving particle (Fig. 15). The product mvp of the momentum and its perpendicular distance from the origin is called the **moment of momentum**, or the **angular momentum**, of the particle about the origin.

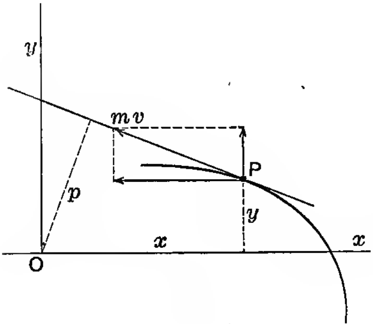

Fig. 15.

It appears from Fig. 15 that we have

$$mvp=mx\frac{dy}{dt}-my\frac{dx}{dt}.$$

The angular momentum is evidently nothing but twice the sectorial velocity multiplied by the mass, just as momentum is linear velocity times mass.

The dynamical meaning of equation (15) can therefore be expressed as follows: *the time-rate of change of angular momen-*

tum about any fixed point is equal to the moment of the resultant force about the same point.

90. The most important case in which the integration in (16) can be performed is the case when

$$xY - yX = 0, \tag{17}$$

which evidently means that the direction of the resultant force F passes through the origin. If this condition be fulfilled, equation (16) reduces to the form

$$mx\frac{dy}{dt} - my\frac{dx}{dt} = c, \tag{18}$$

where c is a constant of integration to be determined from the initial position and velocity.

Kinematically, this equation means that the sectorial velocity remains constant. It can be put into the form

$$\frac{dS}{dt} = \frac{c}{2m} = c',$$

whence, by integration, we find

$$S - S_0 = c'(t - t_0). \tag{19}$$

Hence, *if the acceleration passes constantly through a fixed point, the sector* $S - S_0$ *described about this point in any time* $t - t_0$ *is proportional to this time.*

This is the *principle of the conservation of area* for plane motion.

Dynamically, equation (18) means that *if the resultant force passes constantly through a fixed point, the angular momentum about this point remains constant.* The proposition can also be called the *principle of the conservation of angular momentum.*

If v_0 be the initial velocity, p_0 the perpendicular to v_0 from the fixed point, equation (18) can be written in the form

$$vp = v_0 p_0. \tag{20}$$

91. In the general case of three dimensions any two of the equations of motion,

$$m\frac{d^2x}{dt^2}=X,\quad m\frac{d^2y}{dt^2}=Y,\quad m\frac{d^2z}{dt^2}=Z,$$

can be combined by the method of Art. 87, and we find thus

$$\begin{aligned} my\frac{d^2z}{dt^2}-mz\frac{d^2y}{dt^2}&=\frac{d}{dt}\left(my\frac{dz}{dt}-mz\frac{dy}{dt}\right)=yZ-zY,\\ mz\frac{d^2x}{dt^2}-mx\frac{d^2z}{dt^2}&=\frac{d}{dt}\left(mz\frac{dx}{dt}-mx\frac{dz}{dt}\right)=zX-xZ, \qquad (21)\\ mx\frac{d^2y}{dt^2}-my\frac{d^2x}{dt^2}&=\frac{d}{dt}\left(mx\frac{dy}{dt}-my\frac{dx}{dt}\right)=xY-yX.\end{aligned}$$

The expression $xdy-ydx$ now represents the projection on the xy-plane of the infinitesimal sector described by the radius vector of the particle during the time dt; similarly, $ydz-zdy$ and $zdx-xdz$ are the projections of the same sector on the planes yz and zx, respectively.

92. The right-hand members of the equations (21) are easily seen to represent the moments of the resultant force about the axes of x, y, z, if it be remembered that the moment of a force with respect to an axis is the moment of its projection on a plane perpendicular to the axis about the point of intersection of the axis with the plane (Part II., Art. 213). If the moment of the momentum mv of the particle be defined in the same way, the quantities $mydz/dt-mzdy/dt$, $mzdx/dt-mxdz/dt$, $mxdy/dt-mydx/dt$ are the *moments of momentum*, or, as they are also called, the *angular momenta*, about the three axes of co-ordinates Ox, Oy, Oz.

As the axes are arbitrary, the equations (21) express the statement that *the moment of the resultant about any fixed axis is equal to the time-rate of change of the angular momentum about the same axis.*

93. The most important case of the application of the equations (21) arises when one or more of the conditions

$$yZ-zY=0,\quad zX-xZ=0,\quad xY-yX=0 \qquad (22)$$

are fulfilled. The first of these conditions means that the projection of the resultant force on the yz-plane passes always through a fixed point, viz. the origin of co-ordinates; or what amounts to the same thing, that the resultant force always intersects the axis of x. Similarly, the second condition means that the resultant intersects the axis of y. Hence, if both these conditions are fulfilled, the resultant force passes constantly through a fixed point, the origin of co-ordinates.

It follows that if any two of the conditions (22) are fulfilled, the third must also be fulfilled. This is also evident analytically, as any one of the three equations can be derived from the two others.

94. If the conditions (22) are fulfilled, the integration of the equations (21) gives

$$\begin{aligned} my\frac{dz}{dt}-mz\frac{dy}{dt}&=c_1,\\ mz\frac{dx}{dt}-mx\frac{dz}{dt}&=c_2,\\ mx\frac{dy}{dt}-my\frac{dx}{dt}&=c_3, \end{aligned} \qquad (23)$$

where c_1, c_2, c_3 are constants depending on the initial conditions. These equations express the proposition that *if the resultant force passes constantly through a fixed point, the angular momentum about any axis passing through this point remains constant.*

Multiplying the equations (23) respectively by x, y, z and adding, we find

$$c_1x+c_2y+c_3z=0, \qquad (24)$$

which is the equation of a plane passing through the origin. As the co-ordinates x, y, z of the moving particle fulfil this equation independently of the time, it follows that *the motion is necessarily plane whenever the conditions* (22) *are satisfied.* The constants of integration c_1, c_2, c_3 are evidently proportional to the direction-cosines of the normal to the plane of motion.

95. If the equations (23) be written in the form

$$\frac{1}{2}\frac{ydz-zdy}{dt}=c_1', \quad \frac{1}{2}\frac{zdx-xdz}{dt}=c_2', \quad \frac{1}{2}\frac{xdy-ydx}{dt}=c_3',$$

they show that the projections of the motion on the three co-ordinate planes have constant sectorial velocities c_1', c_2', c_3'; hence, the sectorial velocity of the motion itself is constant, viz.

$$\frac{dS}{dt}=\sqrt{c_1'^2+c_2'^2+c_3'^2}.$$

It follows in this case that the sector $S-S_0$, described during the time $t-t_0$, is proportional to this time:

$$S-S_0=\sqrt{c_1'^2+c_2'^2+c_3'^2}\,(t-t_0). \tag{25}$$

96. Exercises.

(1) A particle of mass m is attracted, according to Newton's law, by a mass m' concentrated at a fixed point O. If x_0, y_0, z_0 be the initial co-ordinates, and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ the initial velocities of the particle, find the equation of the plane in which it moves, and show that this plane passes through O and the initial velocity.

(2) A particle is attracted by n fixed centres, whose forces are directly proportional to the masses of the centres and to the distances from them. Show that there is one position of equilibrium for the particle, and that the motion takes place as if the total mass of all the centres were concentrated at this point. Find also the equation of the plane of the motion.

(3) A particle is acted upon by a central force, *i.e.* by a force whose direction passes through a fixed point, and whose magnitude is a function of the distance from this point, say $F=mf(r)$. Show that the path is a plane curve, and find the equation of the plane of the motion.

(4) The equation (15) can, by Art. 89, be written $d(mvp)/dt=xY-yX$. Show that the two terms of $d(mvp)/dt=mpdv/dt+mvdp/dt$ are equal

respectively to the moments of the tangential and normal components of the resultant force F.

97. The Principle of d'Alembert. Let us consider a particle of mass m moving under the action of any forces $F_1, F_2, \cdots F$, whose resultant is F. The total acceleration j of the particle has the components d^2x/dt^2, d^2y/dt^2, d^2z/dt^2 parallel to the rectangular axes Ox, Oy, Oz. If the forces $F_1, F_2, \cdots F_n$ be imagined removed, a force equal to mj would be required to give the particle the same acceleration j that it had under the action of the forces $F_1, F_2, \cdots F$. This fictitious force, mj, whose components are md^2x/dt^2, md^2y/dt^2, md^2z/dt^2, is called the *effective force*. For the sake of distinction, the forces $F_1, F_2, \cdots F_n$, which actually produce the motion, are called the *impressed forces* (comp. Art. 36).

98. The ordinary equations of motion of a particle,

$$m\frac{d^2x}{dt^2}=X,\quad m\frac{d^2y}{dt^2}=Y,\quad m\frac{d^2z}{dt^2}=Z, \tag{26}$$

where X, Y, Z are the components of the resultant F of the impressed forces, express merely the equality between the effective force mj and the resultant impressed force F. It follows that, *if the reversed effective force* $-mj$, or its components, $-md^2x/dt^2$, $-md^2y/dt^2$, $-md^2z/dt^2$, *be combined with the impressed forces* $F_1, F_2, \cdots F_n$, *we have a system in equilibrium.* This is the fundamental idea of d'Alembert's principle.

99. The reversed effective force, $-mj$, is sometimes called the *force of inertia* of the particle. To understand the idea underlying this expression, imagine the impressed forces to be removed, and then push the particle with the hand so as to give it the same motion that it had under the action of the impressed forces. The pressure of the hand on the particle must at every instant be equal to the resultant F, or to the effective force mj, while the equal and opposite pressure of the particle on the hand represents the force of inertia. It must, however, be clearly understood that this force of inertia, or inertia-resistance, is a force exerted on the hand and not on the particle.

100. Owing to the fact that, by combining with the impressed forces the reversed effective force, we obtain at any given instant a system in equilibrium, it becomes possible to apply to kinetical problems the statical conditions of equilibrium.

Since in the case of a single particle the forces are all concurrent, the conditions of equilibrium are obtained by equating to zero the sums of the components of the forces along each axis. This gives

$$X - m\frac{d^2x}{dt^2} = 0, \quad Y - m\frac{d^2y}{dt^2} = 0, \quad Z - m\frac{d^2z}{dt^2} = 0,$$

and these are the ordinary dynamical equations of motion (see (26), Art. 98).

101. The conditions of equilibrium of a system of forces can also be expressed by means of the principle of virtual work (Part II., Art. 239). Thus, let δx, δy, δz be the components of any virtual displacement δs of the particle; then the principle of virtual work applied to our system of forces gives the single condition

$$\left(-m\frac{d^2x}{dt^2} + X\right)\delta x + \left(-m\frac{d^2y}{dt^2} + Y\right)\delta y + \left(-m\frac{d^2z}{dt^2} + Z\right)\delta z = 0, \quad (27)$$

which is of course equivalent to the three equations (26) on account of the abitrariness of the displacement δs.

The equation (27), which may also be written in the form

$$m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) = X\delta x + Y\delta y + Z\delta z, \quad (28)$$

expresses d'Alembert's principle for a single particle: *for any virtual displacement the sum of the virtual works of the impressed forces is equal to that of the effective force.*

102. The advantage of using the equations of motion in the form given to them by d'Alembert arises mainly from the application of the principle of virtual work which thus becomes possible; this will be seen more clearly later on, in the treatment of constrained motion. For the present it may suffice to notice that, if the actual displacement ds of the particle in its path be selected as the virtual displacement δs, equation (28) becomes

$$m\left(\frac{d^2x}{dt^2}dx+\frac{d^2y}{dt^2}dy+\frac{d^2z}{dt^2}dz\right)=Xdx+Ydy+Zdz. \tag{29}$$

This is the equation of kinetic energy (Art. 71); for the left-hand member is the exact differential $d(\frac{1}{2}mv^2)$ of the kinetic energy, while the right-hand member represents the element of work of the impressed forces.

In the particular, but very common, case of *conservative* impressed forces, the right-hand member is likewise an exact differential dU; hence, in this case a first integration can at once be performed, and we find, as in Art. 75,

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0^2=\int(Xdx+Ydy+Zdz)=U-U_0. \tag{30}$$

103. There is an essential distinction between the principle of d'Alembert on the one hand, and the principles of kinetic energy and of areas on the other. D'Alembert's principle merely gives a convenient form and interpretation to the dynamical equations of motion, through the application of the principle of virtual work; but it does not show how to integrate these equations.

The principle of kinetic energy and the principle of areas are really methods for integrating the equations of motion under certain conditions. If we enquire why these particular methods of combining the differential equations so frequently furnish the solution of physical problems, we are led to the conclusion that the quantities whose exact differentials are introduced by the combination correspond to something really existing in nature. It is thus made probable on purely theoretical grounds that kinetic and potential energy are not mere abstractions, but have an objective reality, and that the conservation of energy is a law of nature.

2. CENTRAL FORCES.

104. We proceed to apply the general principles developed in the preceding articles to the motion of a particle under the action of **central forces.**

The term *central force* is generally understood to imply two conditions, viz. (*a*) that *the direction of the force passes constantly through a fixed point, usually called the centre of force;* and (*b*) that *the magnitude of the force is a function of the distance from the centre alone* (comp. Art. 56).

Let O be the centre of force, P the position of the moving particle at any time t, m the mass of the particle, and $OP=r$ its distance from the centre; then the general expression for a central force F is

$$F=F(r)=mf(r),$$

where the function $F(r)$ represents the law of force, and the function $f(r)$ the law of the acceleration produced by this force in the particle m.

105. The most important special case is that of a force proportional to some power of the distance r, say

$$F(r)=\mu r^n,$$

where μ and n are constants. The constant μ, which represents the value of the force at unit distance from the centre, is often called the *intensity* of the force, or of the centre.

In the case of Newton's law of universal gravitation (Part II., Art. 257) we have $n=-2$, $\mu=\kappa mm'$, where κ is a constant, viz. the acceleration produced by a unit of mass acting on a unit of mass at unit distance, while m is the mass of the attracted particle, and m' that of the attracting centre; that is, Newton's law is expressed by the formula

$$F=\kappa\frac{mm'}{r^2}.$$

106. From the physical point of view, attractions following Newton's law, and indeed, central forces generally, are usually regarded as due to the presence of mass, not only in the moving particle, but also at the centre of force; and the action between these two masses is then a mutual action, being of the nature of a *stress*, *i.e.* consisting of two equal and opposite forces. It follows that what we have called the centre of force is not a fixed point.

It will, however, be shown later (Arts. 150–157) that a simple modification allows us to apply to this case the results deduced on the assumption that the centre is fixed.

Again, the attracting or repelling masses will here be regarded as concentrated at points. It should be remembered that a homogeneous sphere, according to Newton's law, attracts a particle outside of its mass as if the whole mass of the sphere were concentrated at the centre of the sphere (Part II., Arts. 272–276). The attraction of any other mass on a particle can, of course, always be reduced to a single force; but as the particle moves, the direction of this force will not in general pass through a fixed point; such a force is, therefore, not central.

107. If a particle P of mass m be acted upon by a single central force

$$F = mf(r),$$

its acceleration $j = F/m = f(r)$ will pass through the centre of force and be a function of r alone. The problem reduces, therefore, at once to the kinematical problem of *central motion* (Part I., Art. 223). Although the leading ideas of the solution of this problem have been indicated in kinematics (Part I., Arts. 225–234, 237–238), the importance of the subject of central forces demands a restatement in this place of the principal methods in the language of kinetics, and a more complete exposition of some special cases.

108. A particle of mass m acted upon by a single central force $F = mf(r)$ will describe a curvilinear path whenever the initial velocity is different from zero and does not pass through the centre of force (see Art. 56). As shown in kinematics (Part I., Art. 225), the path of the particle, here usually called the **orbit**, is always a plane curve.

Taking the plane of motion as xy-plane and the centre O as origin (Fig. 16), the direction cosines of the force F are $\mp x/r$, $\mp y/r$, the upper sign corresponding to an attractive force, the lower to a repulsion. Hence, the dynamical equations of motion are

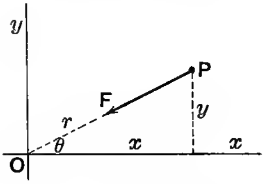

Fig. 16.

$$m\frac{d^2x}{dt^2} = \mp F\frac{x}{r}, \quad m\frac{d^2y}{dt^2} = \mp F\frac{y}{r}. \qquad (1)$$

If $mf(r)$ be substituted for F, the factor m disappears, and the equations become purely kinematical.

109. To avoid the use of the double sign, we shall give the equations in the form corresponding to the more important case of attraction; for a repulsive force it will only be necessary to change throughout the sign of F or $f(r)$. Thus the fundamental equations of motion are (comp. Part I., Art. 226):

$$\frac{d^2x}{dt^2} = -f(r)\frac{x}{r}, \quad \frac{d^2y}{dt^2} = -f(r)\frac{y}{r}. \qquad (2)$$

If polar co-ordinates r, θ (Fig. 16), with the centre as pole, be used, the equations of motion are, since the total acceleration is along the radius vector :

$$j_r \equiv \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -f(r), \quad j_\theta \equiv \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0. \qquad (3)$$

110. Two principal problems present themselves: (*a*) the problem of finding the orbit for a given law of force, and (*b*) the converse problem of determining the law of force, *i.e.* the function $f(r)$, when the orbit is given. The solution of the former problem is effected by obtaining first integrals of the equations of motion from the principle of areas and from the principle of kinetic energy, and by combining these integrals so as to effect a second integration. Formulæ for the solution of the latter problem will be found incidentally.

111. The second of the polar equations (3) gives immediately, if c denote the constant of integration :

$$r^2\frac{d\theta}{dt}=c; \qquad (4)$$

and the meaning of this equation is that *the sectorial velocity is constant* and equal to $\frac{1}{2}c$. The same result can be obtained from the equations (2) by applying the principle of areas (see Arts. 87, 88).

To express the constant c in terms of the initial conditions, let v denote the velocity, p the perpendicular to it from the centre of force, and ψ the angle between the radius vector r and the velocity v, all at the time t (Fig. 17) ; and let the initial values of these quantities, at the time $t=0$, be distinguished by zero-subscripts. Then it follows from the equation (4) that we have (see Art. 89 and Part I., Art. 230)

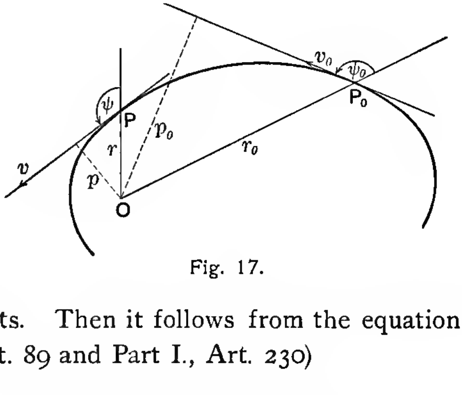

Fig. 17.

$$c=pv=p_0v_0=vr\sin\psi=v_0r_0\sin\psi_0; \qquad (5)$$

i.e. *the velocity is inversely proportional to its perpendicular distance from the centre,* or, as it is sometimes expressed, the moment of the velocity about the centre of force is constant.

112. Another first integral of the equations of motion is obtained by combining the equations (1) according to the principle of kinetic energy (Art. 71). This gives

$$d(\tfrac{1}{2}mv^2)=-Fdr, \quad \text{or} \quad d(\tfrac{1}{2}v^2)=-f(r)dr, \qquad (6)$$

whence $$v^2 = v_0^2 - 2\int_{r_0}^{r} f(r)\,dr; \tag{7}$$

i.e. *the velocity at any distance* r *depends only on this distance* (besides the initial radius vector and velocity), and is independent of the path described, being the same as if the particle had been projected with the initial velocity along the straight line joining the initial position to the centre.

113. To perform the second integration we have only to substitute in (10) for v its value in terms of r and t or r and θ. Now the general expression for the velocity in any curvilinear motion is (Part I., Art. 142)

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right].$$

From these expressions one of the variables θ and t can be eliminated by substituting for $d\theta/dt$ its value c/r^2 from (4); this gives

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{c^2}{r^2} = \frac{c^2}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]. \tag{8}$$

It is often convenient to replace the radius vector r by its reciprocal $u = 1/r$; we then have

$$v^2 = \frac{1}{u^4}\left(\frac{du}{dt}\right)^2 + c^2u^2 = c^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]. \tag{9}$$

114. The formulæ (4) and (7), together with the expressions (8) or (9), contain the complete solution of the two principal problems mentioned in Art. 110. Thus, if the law of force be given, the form of the function $f(r)$ is known, and v can be found from (7) in function of r or u; substituting this value of v in either (8) or (9), we have a differential equation of the first order between r and t, or between r and θ. The integration of the latter equation gives the integral equation of the orbit.

On the other hand, if the equation of the path be given, the expressions (8) or (9) furnish the value of v^2, which, substituted in (6), determines the law of force $f(r)$.

When the equation of the orbit is known, *i.e.* when r is known as a function of θ or *vice versa*, the time t of the motion can be found by integrating equation (4), viz.

$$dt = \frac{1}{c} r^2 d\theta.$$

115. If the second expression for v^2 in (9) be introduced into the differential equation of kinetic energy (6), we find

$$c^2\left(\frac{d^2u}{d\theta^2} + u\right)du = -f(r)dr,$$

or

$$f(r) = c^2u^2\left(\frac{d^2u}{d\theta^2} + u\right). \tag{10}$$

This will generally be found the most convenient form for finding the law of force when the polar equation of the orbit is given. Again, when $f(r)$ is given, the integration of this differential equation of the second order is often more convenient for finding the equation of the orbit than the method indicated in Art. 114.

It may be noted that the important relation (10) can be derived directly from the equations of motion (3), by eliminating t by means of (4) and introducing u for $1/r$. We have

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{c}{r^2}\frac{dr}{d\theta} = -c\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2} = -c\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -c^2u^2\frac{d^2u}{d\theta^2};$$

if these values be substituted into the first of the equations (3), the relation (10) will result.

116. When the equation of the orbit can be expressed conveniently in terms of r and p, as is, for instance, the case for the conic sections, it is of advantage to combine the equation of kinetic energy, $d(\frac{1}{2}v^2) = -f(r)dr$, directly with the equation resulting from the principle of areas, $pv = c$. This gives

$$-f(r) = \frac{1}{2}\frac{d(v^2)}{dr} = \frac{c^2}{2}\frac{d\left(\frac{1}{p^2}\right)}{dr} = -\frac{c^2}{p^3}\frac{dp}{dr}. \tag{11}$$

117. It is easy to see that the methods here explained would apply even to the more general problem *when the force* F, *while passing always through a fixed centre, is not a function of the distance* r *alone, but a function of both co-ordinates* r *and* θ. The principal difference will appear in the impossibility of performing directly the integration indicated in (7).

With $F = mf(r, \theta)$, the equation of kinetic energy is, for attraction,

$$d(\tfrac{1}{2}v^2) = -f(r, \theta)dr;$$

and substituting for v^2 the first of the values given in (8), we find

$$\frac{d^2r}{dt^2} = -f(r, \theta) + \frac{c^2}{r^3}. \qquad (12)$$

This equation shows that the motion relative to the radius vector takes place as if the actual resulting force $F = mf(r, \theta)$ were increased by an additional force mc^2/r^3.

For the law of force we have, as in Art. 115:

$$f(r, \theta) = c^2u^2\left(\frac{d^2u}{d\theta^2} + u\right).$$

118. We proceed to the consideration of some special cases. The most important of these are the case of a force directly proportional to the distance, and that of a force inversely proportional to the square of the distance.

119. Force Proportional to the Distance : $f(r) = \kappa^2 r$. The equations of motions (2) are in this case

$$\frac{d^2x}{dt^2} = \mp\kappa^2 x, \quad \frac{d^2y}{dt^2} = \mp\kappa^2 y,$$

the upper sign holding for attraction, the lower for repulsion. Their solution is very simple, because each equation can be integrated separately. We find, in the case of *attraction*,

$$x = a_1 \cos \kappa t + a_2 \sin \kappa t, \quad y = b_1 \cos \kappa t + b_2 \sin \kappa t,$$

and in the case of *repulsion*,

$$x = a_1 e^{\kappa t} + a_2 e^{-\kappa t}, \quad y = b_1 e^{\kappa t} + b_2 e^{-\kappa t};$$

a_1, a_2, b_1, b_2 being the constants of integration.

120. To find the equation of the orbit, it is only necessary to eliminate t in each case.

In the case of attraction, this elimination can be performed by solving for $\cos\kappa t$, $\sin\kappa t$, squaring and adding. The result is

$$(a_1 y - b_1 x)^2 + (a_2 y - b_2 x)^2 = (a_1 b_2 - a_2 b_1)^2,$$

and this represents an ellipse, since

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2 = (a_1 b_2 - a_2 b_1)^2$$

is always positive. The centre of the ellipse is at the origin, and the lines $a_1 y = b_1 x$, $a_2 y = b_2 x$ are a pair of conjugate diameters.

121. In the case of repulsion, solve for $e^{\kappa t}$ and $e^{-\kappa t}$, and multiply. The resulting equation,

$$(a_1 y - b_1 x)(b_2 x - a_2 y) = (a_1 b_2 - a_2 b_1)^2,$$

represents a hyperbola whose asymptotes are the lines $a_1 y = b_1 x$, $a_2 y = b_2 x$.

122. It is worthy of notice that the more general problem of *the motion of a particle attracted by any number of fixed centres, with forces directly proportional to the distances from these centres*, can be reduced to the problem of Art. 119.

Let x, y, z be the co-ordinates of the particle, r_i its distance from the centre O_i; x_i, y_i, z_i the co-ordinates of O_i; and $-\kappa_i^2 r_i$ the acceleration produced by O_i. Then the x-component of the resultant acceleration is

$$= -\Sigma\kappa_i^2 r_i \cdot \frac{x - x_i}{r_i} = -\Sigma\kappa_i^2(x - x_i) = -x\Sigma\kappa_i^2 + \Sigma\kappa_i^2 x_i;$$

and similar expressions obtain for the y and z components. Hence, the equations of motion are

$$\frac{d^2x}{dt^2} = -x\Sigma\kappa_i^2 + \Sigma\kappa_i^2 x_i, \quad \frac{d^2y}{dt^2} = -y\Sigma\kappa_i^2 + \Sigma\kappa_i^2 y_i, \quad \frac{d^2z}{dt^2} = -z\Sigma\kappa_i^2 + \Sigma\kappa_i^2 z_i.$$

As these expressions are linear in x, y, z, there is one, and only one, point at which the resultant acceleration is zero. Denoting its co-ordinates by $\bar{x}, \bar{y}, \bar{z}$, we have

$$\bar{x} = \frac{\Sigma\kappa_i^2 x_i}{\Sigma\kappa_i^2}, \quad \bar{y} = \frac{\Sigma\kappa_i^2 y_i}{\Sigma\kappa_i^2}, \quad \bar{z} = \frac{\Sigma\kappa_i^2 z_i}{\Sigma\kappa_i^2}.$$

The form of these equations shows that this point of zero acceleration, which is sometimes called the *mean centre*, is the centroid of the centres of force, if these centres be regarded as containing masses equal to κ_i^2. It is evidently a fixed point.

123. By introducing the co-ordinates of the mean centre, we can now reduce the equations of motion to the simple form

$$\frac{d^2x}{dt^2} = -\kappa^2(x - \bar{x}), \quad \frac{d^2y}{dt^2} = -\kappa^2(y - \bar{y}), \quad \frac{d^2z}{dt^2} = -\kappa^2(z - \bar{z}),$$

where $\kappa^2 = \Sigma\kappa_i^2$. Finally, taking the mean centre as origin, we have

$$\frac{d^2x}{dt^2} = -\kappa^2 x, \quad \frac{d^2y}{dt^2} = -\kappa^2 y, \quad \frac{d^2z}{dt^2} = -\kappa^2 z.$$

It thus appears that *the motion of the particle is the same as if there were only a single centre of force*, viz. the mean centre $(\bar{x}, \bar{y}, \bar{z})$, *attracting with a force proportional to the distance from this centre.*

The plane of the orbit is, of course, determined by the mean centre and the initial velocity. The equation of this plane can be found by applying the principle of areas (Art. 94).

124. It is easy to see that most of the considerations of Art. 122 apply even when some or all of the centres *repel* the particle with forces proportional to the distance. It may, however, happen in this case that the mean centre lies at infinity, in which case it can, of course, not be taken as origin.

Simple geometrical considerations can also be used to solve the problem. Thus, in the case of two attractive centres O_1, O_2 (Fig. 18) of equal intensity κ^2, the forces can evidently be represented by the distances $PO_1 = r_1$, $PO_2 = r_2$ of the particle P from the centres. Their resultant is therefore $= 2\,PO$, if O denotes the point midway between O_1 and O_2; and this resultant always passes through this fixed point O, and is proportional to the distance PO from this point.

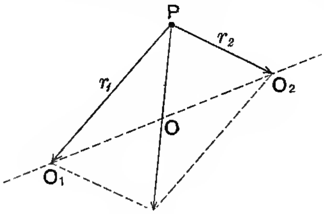

Fig. 18.

125. Exercises.

(1) Determine the constants of integration in Art. 119, if x_0, y_0 are the co-ordinates of the particle at the time $t = 0$ and $\dot{x}_0$, $\dot{y}_0$ the com-

ponents of its velocity v_0 at the same time. The equation of the orbit will assume the form

$$\kappa^2(x_0y-y_0x)^2+(\dot{x}_0y-\dot{y}_0x)^2=(x_0\dot{y}_0-y_0\dot{x}_0)^2$$

for attraction, and

$$\kappa^2(x_0y-y_0x)^2-(\dot{x}_0y-\dot{y}_0x)^2=-(x_0\dot{y}_0-y_0\dot{x}_0)^2$$

for repulsion.

(2) Show that the semi-diameter conjugate to the initial radius vector has the length v_0/κ, where $v_0{}^2=\dot{x}_0{}^2+\dot{y}_0{}^2$. As any point of the orbit can be regarded as initial point, it follows that *the velocity at any point is proportional to the parallel diameter of the orbit.*

(3) Find what the initial velocity must be to make the orbit a circle in the case of attraction, and an equilateral hyperbola in the case of repulsion.

(4) The initial radius vector r_0 and the initial velocity v_0 being given geometrically, show how to construct the axes of the orbit described under the action of a central force (of given intensity κ^2) proportional to the distance from the origin.

(5) A particle describes an ellipse under the action of a central force proportional to the distance; show that the eccentric angle is proportional to the time, and find the corresponding relation for a hyperbolic orbit.

(6) A particle of mass m describes a conic under the action of a central force $F=\mp m\kappa^2r$. Show that the sectorial velocity is $\frac{1}{2}c=\frac{1}{2}\kappa ab$, a and b being the semi-axes of the conic.

(7) In Ex. (6) show that the time of revolution is $T=2\pi/\kappa$, if the conic is an ellipse.

(8) A particle describes a conic under the action of a force whose direction passes through the centre of the conic. Show that the force is proportional to the distance from the centre.

(9) A particle is acted upon by two central forces of the same intensity (κ^2), each proportional to the distance from a fixed centre. Determine the orbit: (a) when both forces are attractive; (b) when both are repulsive; (c) when one is an attraction, the other a repulsion.

(10) A particle of mass m is attracted by two centres O_1, O_2 of equal mass m' and repelled by a third centre O_3, whose mass is $m'' = 2m'$. If the forces are all directly proportional to the respective distances, determine and construct the orbit.

(11) When a particle moves in an ellipse under a force directed towards the centre, find the time of moving from the end of the major axis to a point whose polar angle is θ.

126. Force Inversely Proportional to the Square of the Distance: $f(r) = \mu/r^2$ (Newton's law).

It has been shown in kinematics (Part I., Arts. 229–236) how this law of acceleration can be deduced from Kepler's laws of planetary motion. From Kepler's first law Newton concluded that the acceleration of a planet (regarded as a point of mass m) is constantly directed towards the sun; from the second he found that this acceleration is inversely proportional to the square of the distance. The motion of a planet can therefore be explained on the hypothesis of an attractive force,

$$F = m\frac{\mu}{r^2},$$

issuing from the sun.

The value of μ, which represents the acceleration at unit distance or the so-called intensity of the force, was found to be (Part I., Art. 236; or below, Art. 139)

$$\mu = 4\pi^2 \frac{a^3}{T^2};$$

and as, according to Kepler's third law, the quantity a^3/T^2 has the same value for all the planets, Newton inferred that the intensity of the attracting force is the same for all planets; in other words, that it is one and the same central force that keeps the different planets in their orbits.

127. It was further shown by Newton and Halley that the motions of the comets are due to the same attractive force. The orbits of the comets are generally ellipses of great eccentricity, with the sun at one of the foci. As a comet is within range of observation only while in that portion of its path which lies nearest to the sun, a portion of a parabola, with the same focus and vertex, can be substituted for this portion of the elliptic orbit, at least as a first approximation.

It is also found from observation that the motions of the moons or satellites around the planets follow very nearly Kepler's law. A planet can therefore be regarded as attracting each of its satellites with a force proportional to the mass of the satellite and inversely proportional to the distance.

128. All these facts led Newton to suspect that the force of terrestrial gravitation, as observed in the case of falling bodies on the earth's surface, might be the same as the force that keeps the moon in its orbit around the earth. This inference could easily be tested, since the acceleration g of falling bodies as well as the moon's distance and time of revolution were known.

Let m be the mass of the moon, a the major semi-axis of its orbit, T the time of revolution, r the distance between the centres of earth and moon; then the earth's attraction on the moon is (Art. 126)

$$F = 4\pi^2 m \frac{a^3}{T^2 r^2},$$

or, since the eccentricity of the moon's orbit is so small that the orbit can be regarded as nearly circular,

$$F = 4\pi^2 m \frac{a}{T^2}.$$

On the other hand, the attraction exerted by the earth on a mass m on its surface, *i.e.* at the distance $R = 3963$ miles from the centre, must be

$$F' = mg.$$

Now, if these forces are actually in the inverse ratio of the squares of the distances, we must have

$$\frac{F'}{F} = \frac{a^2}{R^2},$$

or, since the distance of the moon is nearly $= 60\,R$,

$$F' = 60^2 F.$$

Substituting the above values of F and F', we find

$$g = 4\pi^2 \times 60^3 \times \frac{R}{T^2}.$$

With $R = 3963$ miles, $T = 27^d\ 7^h\ 43^m$, this gives

$$g = 32{\cdot}0,$$

a value which agrees sufficiently with the observed value of g, considering the rough degree of approximation used.

129. In this way Newton was finally led to his **law of universal gravitation**, which asserts that *every particle of mass* m *attracts every other particle of mass* m′ *with a force*

$$F = \kappa \frac{mm'}{r^2},$$

where r is the distance of the particles and κ a constant, viz. the acceleration produced by a unit of mass in a unit of mass at unit distance (see Part II., Art. 257, 261–262).

The best proof of this hypothesis as an actual law of physical nature is found in the close agreement of the results of theoretical astronomy based on this law with the observed celestial phenomena.

It may be noticed that, according to this law, the path of a projectile *in vacuo* is only approximately parabolic, the actual path being a very elongated ellipse or hyperbola, one of whose foci is at the earth's centre.

130. Taking Newton's law as a basis, let us now turn to the converse problem of *determining the motion of a particle acted upon by a single central force for which* $f(r) = \mu/r^2$ (problem of planetary motion).

It has been shown in kinematics (Part I., Arts. 239–242) that *if the force be attractive*, the particle will describe a conic section with one of the foci at the centre of force, the conic being an ellipse, parabola, or hyperbola, according as

$$v_0^2 \lesseqgtr \frac{2\mu}{r_0}. \tag{13}$$

If the force be repulsive, the same reasoning will apply, except that μ is then a negative quantity. The orbit is, therefore, in this case always hyperbolic; the branch of the hyperbola that forms the orbit must evidently turn its convex side towards the focus at which the centre of force is situated, since the force always lies on the concave side of the path.

131. To exhibit fully the determination of the constants and the dependence of the nature of the orbit on the initial conditions, a solution somewhat different from that given in kinematics will here be given for the problem of planetary motion in its simplest form.

With $f(r)=\mu/r^2$, the equation of kinetic energy, (7), Art. 112, gives

$$v^2 = v_0^2 - 2\mu\int_{r_0}^{r}\frac{dr}{r^2} = v_0^2 + 2\frac{\mu}{r} - \frac{2\mu}{r_0},$$

or, if the constant of integration be denoted briefly by h and $u=1/r$ be introduced:

$$v^2 = 2\mu u + h, \text{ where } h = v_0^2 - \frac{2\mu}{r_0}. \tag{14}$$

Substituting this expression of v^2 into the equation (9), Art. 113, we find the differential equation of the orbit in the form

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{c^2}(2\mu u + h), \tag{15}$$

or

$$\left(\frac{du}{d\theta}\right)^2 = -\left(u - \frac{\mu}{c^2}\right)^2 + \frac{\mu^2}{c^4} + \frac{h}{c^2}.$$

To integrate, we introduce a new variable u' by putting

$$u - \frac{\mu}{c^2} = u'\sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}};$$

the resulting equation,

$$\left(\frac{du'}{d\theta}\right)^2 = 1 - u'^2, \text{ or } d\theta = \pm\frac{du'}{\sqrt{1-u'^2}},$$

has the general integral

$$\theta - \alpha = \mp \cos^{-1} u', \text{ or } u' = \cos(\theta - \alpha),$$

where α is the constant of integration. The orbit has, therefore, the equation

$$\frac{1}{r} = \frac{\mu}{c^2} + \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}} \cos(\theta - \alpha), \tag{16}$$

which agrees in form with the equation (74) given in kinematics (Part I., Art. 242), excepting the different notation used for the constants.

132. The equation (16) represents a conic section referred to its focus as origin. The general focal equation of a conic is

$$\frac{1}{r} = \frac{1}{l} + \frac{e}{l} \cos(\theta - \alpha), \tag{17}$$

where l is the semi-latus rectum, or parameter, e the eccentricity, and α the angle made with the polar axis by the line joining the focus to the nearest vertex.

In a planetary orbit (Fig. 19), the sun S being at one of the foci, the nearest vertex A is called the *perihelion*, the other vertex A' the *aphelion*, and the angle $\theta - \alpha$ made by any radius vector $SP = r$ with the perihelion distance SA is called the *true anomaly*.

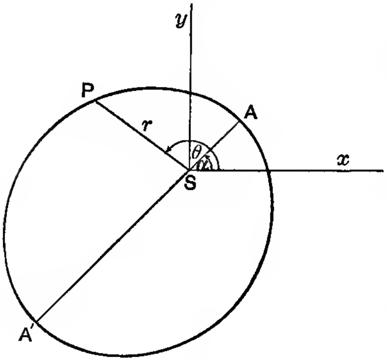

Fig. 19.

Comparing equations (17) and (16), we find, for the determination of the constants:

$$\frac{1}{l} = \frac{\mu}{c^2}, \quad \frac{e}{l} = \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}};$$

hence,

$$l = \frac{c^2}{\mu}, \quad e = \sqrt{1 + \frac{hc^2}{\mu^2}}, \tag{18}$$

or, solving for c and h,

$$c = \sqrt{\mu l}, \quad h = \mu \frac{e^2 - 1}{l}. \tag{19}$$

133. The expression for the eccentricity e in (18) determines the nature of the conic; the orbit is an ellipse, parabola, or hyperbola, according as $e \overset{<}{\underset{>}{=}} 1$; hence, by (18), according as the constant h of the equation of kinetic energy is negative, zero, or positive. Owing to the value of h given in (14), this criterion agrees with the form (13), Art. 130.

It should be observed that it follows from (13) that *the nature* of the conic is independent of the *direction* of the initial velocity.

134. The criterion (13) can be given the following interpretation. Consider a particle attracted by a fixed centre according to Newton's law. If it move in a straight line passing through the centre, the principle of kinetic energy gives for its velocity, at the distance r,

$$v^2 = v_0^2 - 2\mu\int_{r_0}^{r}\frac{dr}{r^2} = \frac{2\mu}{r} + v_0^2 - \frac{2\mu}{r_0};$$

hence, if it start from rest at an infinite distance from the centre, it would acquire the velocity $\sqrt{2\mu/r}$ at the distance r. The criterion (13) is therefore equivalent to saying that *the orbit is an ellipse, a parabola, or a hyperbola, according as the velocity at any point is less than, equal to, or greater than the velocity which the particle would have acquired at that point by falling towards the centre from infinity* (comp. Art. 57).

135. For a *central conic*, whose axes are $2a$, $2b$, we have $l = b^2/a$, $e = \sqrt{a^2 \mp b^2}/a$ (the upper sign relating to the ellipse, the lower to the hyperbola), so that the equations (19) reduce to the following:

$$c = b\sqrt{\frac{\mu}{a}}, \quad h = \mp\frac{\mu}{a}. \tag{20}$$

The latter relation, with the value of h from (14), gives for the major or real semi-axis a:

$$\pm\frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{\mu}; \tag{21}$$

while the former, with the value of c as given in (5), Art. 111, determines the minor or transverse axis b:

$$b = c\sqrt{\frac{a}{\mu}} = r_0 v_0 \sin\psi_0\sqrt{\frac{a}{\mu}}. \tag{22}$$

136. The magnitudes of the axes having thus been found, their directions can be determined by a simple construction which furnishes the second focus.

In the *ellipse*, the focal radii have a constant sum $= 2a$, and lie on the same side of the tangent, making equal angles with it. In the *hyperbola*, they have a constant difference $= 2a$, and lie on opposite sides of the tangent.

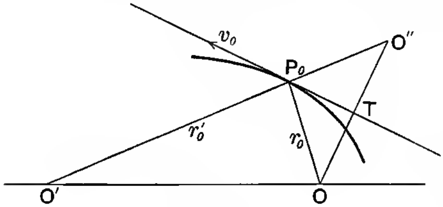

Fig. 20.

Hence, determining the point O'' (Fig. 20), which is symmetrical to the centre of force O with respect to the initial velocity, and drawing the line P_0O'', we have only to lay off on this line from P_0 a length $P_0O' = \pm (2a - r_0)$; then O' is the second focus, which for an elliptic orbit must be taken with O on the same side of the tangent PT, and for a hyperbolic orbit on the opposite side.

137. For a *parabola*, since $e = 1$, we find, from (19),

$$h = 0, \quad l = \frac{c^2}{\mu} = \frac{v_0^2 r_0^2 \sin^2\psi_0}{\mu}. \tag{23}$$

The axis of the parabola is readily found by remembering that the perpendicular let fall from the focus on the tangent bisects the tangent (*i.e.* the segment of the tangent between the point of contact and the axis). Hence, if OT (Fig. 21) be the perpendicular let fall from the centre O on the velocity v_0, it is only necessary to make $TT' = P_0T$, and T' will be a point of the axis. Moreover, the perpendicular let fall from T on OT' will meet the axis at the vertex A of the parabola, so that $OA = \frac{1}{2} l$.

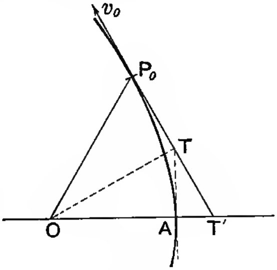

Fig. 21.

138. The relation (21), which must evidently hold at *any* point of the orbit, can be written in the form

$$v^2 = 2\mu\left(\frac{1}{r} \mp \frac{1}{2a}\right), \tag{24}$$

the upper sign relating to the ellipse, the lower to the hyperbola, while for the parabola, the second term in the parenthesis vanishes (since $a = \infty$).

This convenient expression for the velocity in terms of the radius vector might have been derived directly from the fundamental relation (5), $v = c/p$, the first of the equations (19), $c^2 = \mu l$, and the geometrical properties of the conic sections ($r \pm r' = 2a$, $pp' = b^2$, $p'r = pr'$, where r, r' are the focal radii, and p, p' the perpendiculars let fall from the foci on the tangent). The proof is left to the student.

139. Time. In the case of an elliptic orbit, the time T of a complete revolution, usually called the **periodic time**, is found by remembering that the sectorial velocity is constant and $= \frac{1}{2}c$ (Art. 111), whence

$$T = \frac{2\pi ab}{c},$$

or, by (20),

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = \frac{2\pi}{n}. \qquad (25)$$

The constant

$$n = \sqrt{\frac{\mu}{a^3}},$$

which evidently represents the mean angular velocity in one revolution, is called the **mean motion** of the planet. It should be noticed that it depends not only on the intensity of the force, but also on the major axis of the orbit, while in the case of a force directly proportional to the distance it is independent of the size of the orbit (see Art. 125, Ex. 7).

The periodic time T and the major axis a of a planetary orbit determine the intensity μ of the force

$$\mu = 4\pi^2\frac{a^3}{T^2}, \qquad (26)$$

whence

$$F = mf(r) = m\frac{\mu}{r^2} = 4\pi^2 m\frac{a^3}{T^2 r^2}, \qquad (27)$$

where m is the mass of the planet.

140. To find generally the time t in terms of θ or r, we can, of course, proceed as indicated in Art. 114; but the resulting expressions are somewhat complicated, and it is best to introduce the eccentric angle ϕ of the ellipse as a new variable, and to express t, r, and θ in terms of ϕ. In astronomy, the polar angle θ is known as the *true anomaly*, and the eccentric angle ϕ as the *eccentric anomaly*.

141. The relation of the eccentric angle ϕ to the polar co-ordinates r, θ will appear from Fig. 22, in which P is the position of the planet at the time t, P' the corresponding point on the circumscribed circle, $\measuredangle AOP = \theta$ the true anomaly, and $\measuredangle ACP' = \phi$ the eccentric anomaly. The focal equation of the ellipse

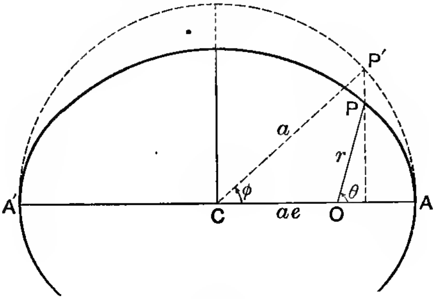

Fig. 22.

$$r = \frac{l}{1 + e\cos\theta} = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

gives $r + er\cos\theta = a - ae^2$; and the figure shows that $r\cos\theta = a\cos\phi - ae$; hence,

$$r = a(1 - e\cos\phi), \text{ or } a - r = ae\cos\phi. \tag{28}$$

Equating this value of r to that given by the polar equation of the ellipse, we have

$$1 - e\cos\phi = \frac{1 - e^2}{1 + e\cos\theta}, \quad \text{or} \quad \cos\theta = \frac{\cos\phi - e}{1 - e\cos\phi}.$$

A more symmetrical form can be given to this relation by computing

$$1 - \cos\theta \equiv 2\sin^2\frac{\theta}{2} = (1 + e)\frac{1 - \cos\phi}{1 - e\cos\phi},$$

$$1 + \cos\theta \equiv 2\cos^2\frac{\theta}{2} = (1 - e)\frac{1 + \cos\phi}{1 - e\cos\phi};$$

whence, by division, $$\tan\frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}}\tan\frac{\phi}{2}. \tag{29}$$

142. To find t in terms of r, we have only to substitute in (24) for v^2 its value from (8), Art. 113, and to integrate the resulting differential equation

$$\left(\frac{dr}{dt}\right)^2 + \frac{c^2}{r^2} = \frac{2\mu}{r} - \frac{\mu}{a}.$$

As, by (20), Art. 135, $c^2 = \mu b^2/a = \mu a(1 - e^2)$, this equation becomes

$$r^2\left(\frac{dr}{dt}\right)^2 = \frac{\mu}{a}\left[a^2e^2 - (a - r)^2\right],$$

or

$$dt = \sqrt{\frac{a}{\mu}}\frac{rdr}{\sqrt{a^2e^2 - (a - r)^2}}.$$

The integration is easily performed by introducing the eccentric angle ϕ as variable by means of (28); this gives

$$dt = \sqrt{\frac{a}{\mu}} \cdot a(1 - e\cos\phi)d\phi.$$

If the time be counted from the perihelion passage of the planet, we have $t = 0$ when $r = a - ae$, *i.e.* when $\phi = 0$; hence, putting $\sqrt{\mu/a^3} = n$, as in Art. 139, we find

$$nt = \phi - e\sin\phi. \tag{30}$$

This relation is known as *Kepler's equation;* the quantity nt is called the *mean anomaly.*

143. Kepler's equation (30) can be derived directly by considering that the ellipse APA' (Fig. 22) can be regarded as the projection of the circle $AP'A'$, after turning this circle about AA' through an angle $= \cos^{-1}(b/a)$. For it follows that the elliptic sector AOP is to the circular sector AOP' as b is to a. Now, for the circular sector we have

$$AOP' = ACP' - OCP' = \tfrac{1}{2}a^2\phi - \tfrac{1}{2}ae \cdot a\sin\phi = \frac{a^2}{2}(\phi - e\sin\phi);$$

hence, the elliptic sector described in the time t is

$$AOP = \frac{b}{a} \cdot AOP' = \frac{ab}{2}(\phi - e\sin\phi).$$

The sectorial velocity being constant by Kepler's first law, we have

$$\frac{AOP}{t} = \frac{\pi ab}{T};$$

hence,

$$t = \frac{T}{2\pi}(\phi - e\sin\phi),$$

and this agrees with (30) since, by (25), $2\pi/T = n$.

144. Kepler's equation (30) gives the time as a function of ϕ; by means of (28), it establishes the relation between t and r; by means of (29), it connects t with θ. It is, however, a transcendental equation and cannot be solved for ϕ in a finite form.

For orbits with a small eccentricity e, an approximate solution can be obtained by writing the equation in the form

$$\phi = nt + e \sin \phi,$$

and substituting under the sine for ϕ its approximate value nt:

$$\phi = nt + e \sin nt. \tag{31}$$

This amounts to neglecting terms containing powers of e above the first power.

Substituting this value of ϕ in (28), we have with the same approximation

$$r = a(1 - e \cos nt). \tag{32}$$

To find θ in terms of t, we have, from the equation of the ellipse, $r = a(1 - e^2)(1 + e \cos \theta)^{-1} = a(1 - e \cos \theta)$, neglecting again terms in e^2; hence, $r^2 = a^2(1 - 2e \cos \theta)$. Substituting this value in the equation of areas, $r^2 d\theta = c dt = \sqrt{\mu a(1 - e^2)}\, dt$, we find

$$(1 - 2e \cos \theta) d\theta = \sqrt{\frac{\mu}{a^3}}\, dt = n dt;$$

whence, by integration, since $\theta = 0$ for $t = 0$,

$$\theta - 2e \sin \theta = nt,$$

or finally,

$$\theta = nt + 2e \sin nt. \tag{33}$$

Thus we have in (31), (32), (33) approximate expressions for ϕ, r, and θ directly in terms of the time. The quantity $2e \sin nt$, by which the true anomaly θ exceeds the mean anomaly nt, is called the *equation of the centre.*

145. Exercises.

(1) A particle describes an ellipse under the action of a central force. Determine the law of force by means of formula (11), Art. 116: (*a*) when the centre of force is at the centre of the ellipse; (*b*) when it is at a focus.

(2) A particle is attracted by a fixed centre according to Newton's law. What must be the initial velocity if the orbit is to be circular?

(3) A number of particles are projected, from the same point in the field of a force following Newton's law, with the same velocity, but in different directions. Show that the periodic times are the same for all the particles.

(4) The mean distance of Mars from the sun being 1.5237 times that of the earth, what is the time of revolution of Mars about the sun?

(5) A particle describes a conic under the action of a central force following Newton's law; if the intensity μ of the force be suddenly changed to μ', what is the effect on the orbit?

(6) In Ex. (5), if the original orbit was a parabola and the intensity be doubled, what is the new orbit?

(7) Regarding the moon's orbit about the earth as circular, what would it become: (*a*) if the earth's mass were suddenly doubled? (*b*) if it were reduced to one-half?

(8) In Ex. (5), determine the effect on the major semi-axis (or "mean distance") a and on the periodic time T, of a *small* change in the intensity μ of the force.

(9) If the mass of the sun be suddenly increased by a small amount while the earth is at the end of the minor axis of its orbit, what would be the effect on the earth's mean distance and on the period of revolution T?

(10) Find the equation of the hodograph of planetary motion, derive from it the expression for the velocity in terms of the radius vector, and show that the velocity is a maximum in perihelion and a minimum in aphelion.

(11) Show that the greatest velocity of a planet in its orbit about the sun is to its least velocity as $\sqrt{1+e}$ is to $\sqrt{1-e}$; and find this ratio for the earth, whose orbit has the eccentricity $e = 0.01677120$.

(12) Find the time exactly as a function of θ, for a parabolic orbit.

146. Force any Function of the Distance. The general methods have been given in Arts. 108–116. The equation of energy, (6), Art. 112, gives, with $u = 1/r$,

$$v^2 = 2\int\frac{f(u)\,du}{u^2} + h; \tag{34}$$

hence, substituting for v its value from (9), Art. 113, we find, for the differential equation of the orbit,

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2}{c^2}\int\frac{f(u)\,du}{u^2}+h'. \tag{35}$$

As it is often difficult or impossible to perform the integration in finite form, it is of importance to determine the apses and apsidal distances of the orbit.

147. An **apse** is a point of the orbit at which the velocity is at right angles to the radius vector drawn from the centre of force; the length of the radius vector of an apse is called the **apsidal distance**, and its direction an **apsidal line**.

The importance of the apsidal lines lies in the fact that they are lines of symmetry of the orbit, while the apsidal distances are maximum or minimum values of the radius vector. This will be seen from the following considerations.

By the above formula (34) the velocity is a function of the radius vector alone; and by (5), Art. 111, since $\sin\psi=c/vr$, the angle ψ between radius vector and velocity is also a function of the radius vector alone. It follows that, if the velocity be reversed in direction at any point of the orbit, the same orbit will be described in the opposite sense; and as at an apse the velocity is perpendicular to the apsidal line, the two portions of the orbit on opposite sides of an apsidal line must be symmetrical with respect to this line.

148. The condition for an apse is therefore

$$\frac{du}{d\theta}=0.$$

Substituting this value in the above equation (35), the apsidal distances $1/u$ can be found by solving the equation for u. The value of $du/d\theta$ should also change sign as the particle passes through the apse.

If the law of force is given as a single-valued function $f(u)$, there can exist only two different apsidal distances (although there may be any number of apses). The angle between these two different apsidal distances is called the **apsidal angle**.

149. Exercises.

(1) Find the law of force when the equation of the orbit is $r^n = a \cos n\theta + b$, and investigate the particular cases $n = -1$, $n = -2$, $n = 1$, $n = 2$.

(2) A particle moves in a circle under the action of a single force of constant direction; determine the law of force and discuss the motion.

(3) Find the law of the central force directed to the origin under whose action a particle will describe the following curves: (*a*) the spiral of Archimedes $r = a\theta$; (*b*) the hyperbolic spiral $\theta r = a$; (*c*) the logarithmic or equiangular spiral $r = ae^{a\theta}$; (*d*) the curve $r = a \cos n\theta$.

(4) A particle moves in a circle under the action of a central force directed towards a point on the circumference; find the law of force.

(5) A particle is acted upon by a force perpendicular to a given plane and inversely proportional to the cube of the distance from the plane. Determine its motion.

(6) A particle moves in a semi-ellipse under the action of a force perpendicular to the axis joining the ends of the semi-ellipse. Determine the law of force and the velocity at the ends.

3. THE PROBLEM OF TWO BODIES.

150. In the preceding discussion of the motion of a particle under the action of a central force, it has been assumed that the centre of force is fixed. In the applications of the theory of central forces this assumption is in general not satisfied. Thus, in considering the motion of a planet around the sun, the force of attraction is, according to Newton's law of universal gravitation (Art. 129), regarded as due to the presence of a mass M at the centre (sun), and of a mass m at the attracted point (planet); and the action between these two masses is a mutual action, being of the nature of a *stress*, *i.e.* consisting of two equal and opposite forces, each equal to

$$F = \kappa \frac{mM}{r^2}.$$

Hence, the mass m of the planet attracts the mass M of the sun with precisely the same force with which the mass M of the sun attracts the mass m of the planet. The attraction affects, therefore, the motions of both bodies.

151. The *accelerations* produced by the two forces are, of course, not equal. Indeed, the acceleration $F/m = \kappa M/r^2$, produced in the planet by the sun, is very much greater than the acceleration $F/M = \kappa m/r^2$, produced by the planet in the sun; for the mass of even the largest planet (Jupiter) is less than one thousandth of that of the sun. The assumption of a fixed centre can therefore be regarded as a first approximation in the problem of the motion of a planet about the sun.

In the case of the earth and moon, the difference of the masses is not so great, the mass of the moon being nearly one eightieth of that of the earth.

It can, however, be shown that the results deduced on the assumption of a fixed centre can, by a simple modification, be made available for the solution of *the general problem of the motions of two particles of masses*, m, M, *subject to no forces besides their mutual attraction.* In astronomy, this is called the **problem of two bodies.** In the solution below we assume the attraction to follow Newton's law of the inverse square of the distance. It will be convenient to speak of the two particles, or bodies, as planet (m) and sun (M).

152. With regard to any fixed system of rectangular axes, let x, y, z be the co-ordinates of the planet (m), at the time t; x', y', z' those of the sun (M), at the same time; so that for their distance r we have

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

Then the equations of motion of the planet are

$$\begin{aligned} m\frac{d^2x}{dt^2} &= \kappa\frac{Mm}{r^2}\cdot\frac{x'-x}{r}, \\ m\frac{d^2y}{dt^2} &= \kappa\frac{Mm}{r^2}\cdot\frac{y'-y}{r}, \\ m\frac{d^2z}{dt^2} &= \kappa\frac{Mm}{r^2}\cdot\frac{z-z}{r}; \end{aligned} \tag{1}$$

while the equations of motion of the sun are

$$\begin{aligned} M\frac{d^2x'}{dt^2} &= \kappa\frac{mM}{r^2}\cdot\frac{x-x'}{r}, \\ M\frac{d^2y'}{dt^2} &= \kappa\frac{mM}{r^2}\cdot\frac{y-y'}{r}, \\ M\frac{d^2z'}{dt^2} &= \kappa\frac{mM}{r^2}\cdot\frac{z-z'}{r}. \end{aligned} \tag{2}$$

153. By adding the corresponding equations of the two sets, we find

$$\frac{d^2}{dt^2}(mx+Mx')=0, \quad \frac{d^2}{dt^2}(my+My')=0, \quad \frac{d^2}{dt^2}(mz+Mz')=0.$$

If it be remembered that the centroid of the two masses m, M has the co-ordinates

$$\bar{x}=\frac{mx+Mx'}{m+M}, \quad \bar{y}=\frac{my+My'}{m+M}, \quad \bar{z}=\frac{mz+Mz'}{m+M},$$

it appears that these equations can be written in the form

$$\frac{d^2\bar{x}}{dt^2}=0, \quad \frac{d^2\bar{y}}{dt^2}=0, \quad \frac{d^2\bar{z}}{dt^2}=0;$$

in words: *the acceleration of the common centroid of planet and sun is zero;* i.e. *this centroid moves with constant velocity in a straight line.*

It may be noticed that this result is merely a special case of a more general proposition to be proved hereafter, viz. that the centroid of any system acted upon by no forces external to the system moves uniformly in a straight line (Art. 381).

154. The integration of the equations (1) would give the absolute path of the planet. But the constants could not be determined, because the absolute initial position and velocity of the planet are, of course, not known. The same holds for the absolute path of the sun. All we can do is to determine the *relative* motion, and we proceed to find the motion of the planet relative to the sun.

Taking the sun's centre as new origin for parallel axes, we have for the co-ordinates ξ, η, ζ of the planet in this new system,

$$\xi = x - x', \quad \eta = y - y', \quad \zeta = z - z'.$$

Now, dividing the equations (1) by m, the equations (2) by M, and subtracting the equations of set (2) from the corresponding equations of set (1), we find for the relative accelerations of the planet

$$\begin{aligned} \frac{d^2\xi}{dt^2} &= -\kappa\frac{M+m}{r^2}\cdot\frac{\xi}{r}, \\ \frac{d^2\eta}{dt^2} &= -\kappa\frac{M+m}{r^2}\cdot\frac{\eta}{r}, \\ \frac{d^2\zeta}{dt^2} &= -\kappa\frac{M+m}{r^2}\cdot\frac{\zeta}{r}. \end{aligned} \tag{3}$$

The form of these equations shows that *the relative motion of the planet with respect to the sun is the same as if the sun were fixed and contained the mass* M + m. Thus the problem is reduced to that of a fixed centre, the only modification being that the mass of the centre M should be increased by that of the attracted particle m.

155. This result can also be obtained by the following simple consideration. The *relative* motion of the planet with respect to the sun would obviously not be altered if geometrically equal accelerations were applied to both. Let us, therefore, subject each body to an additional acceleration equal and opposite to the actual acceleration of the sun (whose components are obtained by dividing the equations (2) by M).

Then the sun will be reduced to equilibrium, while the resulting acceleration of the planet, which is its relative acceleration with respect to the sun, will evidently be the sum of the acceleration exerted on it by the sun, and the acceleration exerted on the sun by the planet. This is just the result expressed by the equations (3).

156. It can here only be mentioned in passing that, while the problem of two bodies thus leads to equations that can easily be integrated, *the problem of three bodies* is one of exceeding difficulty, and has been solved only in a few very special cases. Much less has it been possible to integrate the $3n$ equations of the problem of n bodies.

157. According to the equations (3), the first and second laws of Kepler can be said to hold for the *relative* motion of a planet about the sun (or of a satellite about its primary). The third law of Kepler requires some modification, since the intensity of the centre μ should not be $=\kappa M$, but $=\kappa(M+m)$. Thus we have, by (26), Art. 139,

$$\mu=\kappa(M+m)=4\pi^2\frac{a^3}{T^2};$$

in other words, the quotient a^3/T^2 is not independent of the mass m of the planet.

Thus, if m_1, m_2 be the masses of two planets, a_1, a_2 the major semi-axes of their orbits, and T_1, T_2 their periodic times, we have

$$\frac{a_1^3/T_1^2}{a_2^3/T_2^2}=\frac{M+m_1}{M+m_2}=\frac{1+m_1/M}{1+m_2/M}.$$

This quotient is approximately equal to one if M is very large in comparison with both m_1 and m_2; hence, for the orbits of the planets about the sun, Kepler's law is very nearly true.

158. Exercises.

(1) Two particles of masses m_1, m_2 attract each other with a force which is any function of the distance r between them, say $F=m_1m_2f(r)$. Show that their common centroid moves uniformly in a straight line, and find the equations of this line.

(2) In Ex. (1), write out the equations for the relative motion of either particle with respect to the common centroid.

159. The theory of central forces is treated with considerable elaboration in most works on theoretical mechanics; a few references only will here be given: P. APPELL, *Traité de mécanique rationnelle*, Paris, Gauthier-Villars, 1893, Vol. I., pp. 354–405; B. WILLIAMSON and F. A. TARLETON, *An elementary treatise on dynamics*, London, Longmans (New York, Appleton), 2d edition, 1889, pp. 147–205; P. G. TAIT and W. J. STEELE, *A treatise on dynamics of a particle*, London and New York, Macmillan, 6th edition, 1889, pp. 113–166; W. H. BESANT, *A treatise on dynamics*, Cambridge, Bell (New York, Macmillan), 2d edition, 1893, pp. 120–166, and 267–275; W. WALTON, *A collection of problems in illustration of the principles of theoretical mechanics*, Cambridge, Bell, 3d edition, 1876, pp. 248–297. All these works contain numerous examples for practice. The theory of planetary motion is, of course, treated in works on theoretical astronomy. The student will also consult with advantage: W. SCHELL, *Theorie der Bewegung und der Kräfte*, Leipzig, Teubner, 2d edition, Vol. I., 1879, pp. 373–387; E. BUDDE, *Allgemeine Mechanik der Punkte und starren Systeme*, Berlin, Reimer, Vol. I., 1890, pp. 170–181; B. PRICE, *A treatise on analytical mechanics*, Oxford, Clarendon Press (New York, Macmillan), Vol. I. (= Vol. III. of *A treatise on infinitesimal calculus*), 2d edition, 1868, pp. 508–574; O. RAUSENBERGER, *Lehrbuch der analytischen Mechanik*, Leipzig, Teubner, Vol. I., 1888, pp. 32–102, where the problem of planetary motion is very fully discussed; T. DESPEYROUS, *Cours de mécanique, avec des notes par G. Darboux*, Paris, Hermann, 1884, Vol. I., pp. 336–369, 427–440, and Vol. II., pp. 38–57, 461–466; and others.

IV. *Constrained Motion.*

I. INTRODUCTION.

160. It has been shown, in the preceding sections, that the motion of a free particle is fully determined if all the forces acting upon the particle, as well as the so-called initial conditions, are given. The motion of a particle may, however, depend not only on given forces, but on other conditions not directly expressed in terms of forces. The motion is then said to be *constrained.*

Some of the more important forms of constraint have been considered in Part II., Arts. 218–225. To mention some more concrete examples: a heavy particle sliding down a smooth inclined plane is subject not only to the force of gravity, but also to the condition that it cannot pass through the plane; a railway train running on the rails, a piece of machinery sliding in a groove or between guides, can, for many purposes, be regarded as a particle constrained to a curve; the bob of a pendulum, a stone attached to a cord and swung around by the hand, may be regarded as constrained to a surface.

161. Sometimes these constraining conditions can be easily replaced by forces. Thus, in the first illustration above, the condition that the particle cannot pass through the inclined plane can be expressed by introducing the reaction of the plane, *i.e.* a force acting on the particle at right angles to the plane, so as to prevent it from passing through the plane. Similarly, in the case of the stone attached to the cord, we may imagine the cord cut and its tension introduced so as to replace the condition by a force.

Whenever the constraints to which a particle is subjected can thus be expressed by means of forces, these forces can be combined with the other impressed forces, and then, of course, the equations of motion for a free particle can be applied.

Thus, let X', Y', Z' be the components of the resultant of all the constraints; X, Y, Z those of the resultant of all the other impressed forces. Then the equations of motion are:

$$m\frac{d^2x}{dt^2}=X+X', \quad m\frac{d^2y}{dt^2}=Y+Y', \quad m\frac{d^2z}{dt^2}=Z+Z'. \qquad (1)$$

162. It must, however, be noticed that the reactions representing the constraints, such as the tension of the string in the example referred to, are generally not given beforehand. Moreover, the constraints are often expressed more conveniently by conditional equations. Thus, if the motion of a particle be restricted to a surface, the equation of this surface, say

$$\phi(x, y, z)=0, \qquad (2)$$

may be given as a constraining condition to be fulfilled by the co-ordinates of the moving particle.

163. As a particle has but three degrees of freedom, it can be subjected to only one or two conditions of the form (2). One such condition confines it to a surface; two to the curve of intersection of the two surfaces represented by the two conditional equations; three conditions would evidently prevent it entirely from moving.

164. The curve or surface to which a particle is constrained may vary its position and even its shape in the course of time. In this case the conditional equations, referred to fixed axes, will contain not only the co-ordinates, but also the time. That is, they will be of the more general form

$$\phi(x, y, z, t)=0. \qquad (3)$$

165. To constrain a particle *completely* to a surface, we may imagine it confined between two infinitely near impenetrable surfaces. The complete constraint to a curve might be realized by confining the particle to an infinitely narrow tube having the shape of the curve, or by regarding it as a ring sliding along a wire.

In many cases, however, the constraint is not complete, but only partial, or one-sided. Thus, the rails compelling the train to move in a definite curve do not prevent its being lifted vertically out of this curve, nor does the cord that confines the motion of the stone to a sphere prevent it from moving towards the inside of the spherical surface.

While complete constraints are generally expressed by equations, one-sided constraints should properly be expressed by inequalities. Thus, in the case of the stone, the condition is really that its distance r from the hand is not greater than the length l of the cord, *i.e.*

$$r \leqq l;$$

but as soon as r becomes less than l, the constraining action ceases, and the stone becomes free. It is, therefore, in general sufficient to consider conditional *equations*; but the nature of the constraint, whether complete or partial, must be taken into account to determine when and where the constraint ceases to exist.

166. We now proceed to consider separately the motion of a particle constrained to a fixed curve and that of a particle constrained to a fixed surface. After these special cases, the general problem of motion on a movable curve or surface will be discussed.

2. MOTION ON A FIXED CURVE.

167. The condition that a particle should move on a given fixed curve can always be replaced by introducing a single additional force F' called the *constraining force*, or the *constraint*. An example will best show how this force can be determined.

Let us consider a particle of mass m, subject to the force of gravity $F = mg$ alone; in general it will describe a parabola whose equation can be found if the initial conditions are known. To compel the particle to describe some other curve, say a vertical circle, a constraining force F' (Fig. 23) must be introduced such that the resultant R of F and F' shall produce the acceleration required for motion in the circle. Thus, for instance, for *uniform* motion in a circle the resulting acceleration must be directed towards the centre and must be $=\omega^2 a$, if a is the radius and ω the constant angular velocity. We have, therefore, in this case $R=m\omega^2 a$ along the radius, $F=mg$ vertically downwards; and hence, denoting by θ the angle made by the radius CP with the vertical (Fig. 23),

$$F'^2=F^2+R^2+2\,FR\cos\theta$$

$$=m^2(g^2+\omega^4a^2+2\,g\omega^2a\cos\theta).$$

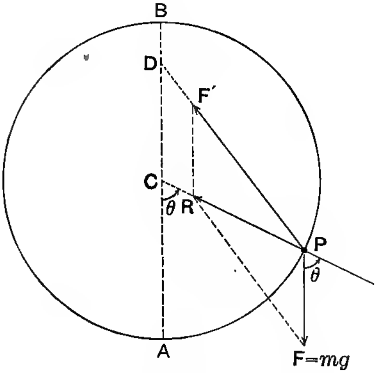

Fig. 23.

The constraint F', which is thus seen to vary with the angle θ, can be resolved into a tangential component F_t' and a normal component F_n'. As in our problem the velocity is to remain of constant magnitude, the tangential constraint must just counterbalance the tangential component $F_t=mg\sin\theta$ of gravity. The normal constraint F_n' not only counterbalances the normal component $F_n=mg\cos\theta$ of gravity, but also furnishes the centripetal force $R=m\omega^2 a$ required for motion in the circle; *i.e.*

$$F_n'=R+F\cos\theta=m(\omega^2a+g\cos\theta).$$

168. In the general case of a particle of mass m acted upon by any given forces and constrained to any fixed curve, it is convenient to resolve both the resultant F of the given forces and the constraint F' along the tangent and the normal plane.

The equations of motion (see Art. 67) can then be written in the form

$$m\frac{dv}{dt}=F_t-F_t',$$

$$m\frac{v^2}{\rho}=\text{resultant of } F_n \text{ and } F_n',$$

where v is the velocity and ρ the radius of curvature of the path at the time t. It should be noticed that the components F_n and F_n', though both situated in the normal plane, do not in general have the same direction. But in the important special case of plane motion, *i.e.* when the path is a plane curve and the resultant F of the given forces lies in this plane, F_n and F_n' are both directed along the radius of curvature so that the right-hand member of the second equation becomes the sum or difference of F_n and F_n'.

169. The normal component F_n' of the constraining force is generally denoted by the letter N and is called the *resistance* or *reaction* of the curve; a force $-N$, equal and opposite to this reaction, represents the *pressure* exerted by the particle on the curve.

The tangential component F_t' of the constraint will exist only when the constraining curve is rough, *i.e.* offers frictional resistance; we have then, denoting the coefficient of friction by μ,

$$F_t'=\mu N.$$

We shall therefore write the equations of motion as follows:

$$m\frac{dv}{dt}=F_t-\mu N, \qquad (1)$$

$$m\frac{v^2}{\rho}=\text{resultant of } F_n \text{ and } N. \qquad (2)$$

170. The normal component, mv^2/ρ, of the effective force is sometimes called the *centripetal force* (see Art. 67); it is directed along the principal normal of the path towards the centre of curvature. A force equal and opposite to this centripetal force, *i.e.* $= -mv^2/r$, is called **centrifugal force.** It should be noticed that this is a force exerted not *on* the moving particle, but *by* it.

It appears from equation (2) that the normal reaction N is the resultant of the centripetal force mv^2/ρ and the reversed normal component of the given forces, $-F_n$. Changing all the signs, we can express the same thing by saying that *the pressure on the curve*, $-\mathrm{N}$, *is the resultant of the centrifugal force*, $-\mathrm{mv}^2/\rho$, *and of the normal component* F_n *of the given forces.*

If, in particular, this normal component F_n is zero, the pressure on the curve is equal to the centrifugal force. This case is of frequent occurrence. Thus, if a small stone attached to a cord be whirled around rapidly, the action of gravity on the stone can be neglected in comparison with the centripetal force due to rotation; hence the centrifugal force measures approximately the tension of the string, and may cause it to break. Again, when a railway train runs in a curve, the centrifugal force produces the horizontal pressure on the rails, which tends to displace and deform the rails.

171. It may happen that at a certain time t the pressure $-N$ vanishes. If the constraint be complete (Art. 165), this would merely indicate that the pressure in passing through zero inverts its sense. If, however, the constraint be one-sided, the consequence will be that the particle at this time leaves the constraining curve; for at the next moment the pressure will be exerted in a direction in which the particle is free to move.

Now N vanishes when its components $-F_n$ and mv^2/ρ become equal and opposite. The conditions under which the particle would leave the curve are, therefore, that the resultant F of the given forces should lie in the osculating plane of the path, and that $F_n = mv^2/\rho$.

172. To obtain the equations of motion expressed in rectangular Cartesian co-ordinates, let X, Y, Z be the components of the resultant F of the given forces, and N_x, N_y, N_z those of the normal reaction N of the curve. If there be friction, the frictional resistance μN, being directed along the tangent to the path opposite to the sense of the motion, has the direction cosines $-dx/ds$, $-dy/ds$, $-dz/ds$, so that the components of the force of friction are $-\mu N dx/ds$, $-\mu N dy/ds$, $-\mu N dz/ds$. The general equations of motion are, therefore,

$$
\begin{aligned}
m\frac{d^2x}{dt^2} &= X + N_x - \mu N\frac{dx}{ds},\\
m\frac{d^2y}{dt^2} &= Y + N_y - \mu N\frac{dy}{ds},\\
m\frac{d^2z}{dt^2} &= Z + N_z - \mu N\frac{dz}{ds}.
\end{aligned}
\qquad (3)
$$

If the acceleration of the particle be zero, the left-hand members are all $=0$, and the equations reduce to the conditions of equilibrium of a particle on a curve, as given in Statics (Part II., p. 138, (14)).

In addition to the equations (3) we have of course the equations of the curve, say

$$\phi(x, y, z) = 0, \quad \psi(x, y, z) = 0, \qquad (4)$$

and the relations

$$N^2 = N_x^2 + N^{\,2} + N_z^2, \qquad (5)$$

$$N_x\frac{dx}{ds} + N_y\frac{dy}{ds} + N_z\frac{dz}{ds} = 0, \qquad (6)$$

the latter expressing that N is perpendicular to the element ds of the path.

173. Multiplying the equations (3) by dx, dy, dz, and adding, we find the equation of kinetic energy

$$d(\tfrac{1}{2}mv^2) = X dx + Y dy + Z dz - \mu N ds. \qquad (7)$$

This relation might have been obtained directly from the consideration that for a displacement ds along the fixed curve the normal reaction N does no work, while the work of friction is $-\mu N ds$.

174. Exercises.

(1) Show that when the given forces are zero and there is no friction, the particle moves uniformly on the curve, and the pressure on the curve is proportional to the curvature of the path.

(2) *A particle of mass* m *moves down a straight line inclined to the horizon at an angle θ, under the action of gravity alone.*

(*a*) If there be no friction, we have by Art. 169, since $\rho=\infty$ (see Fig. 24).

$$m\frac{dv}{dt}=mg\sin\theta,$$

$$0=mg\cos\theta-N.$$

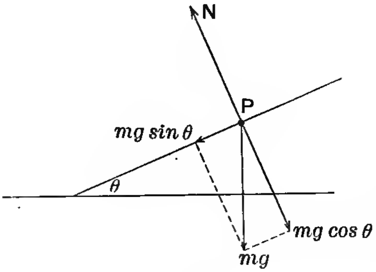

Fig. 24.

The first of these equations is the same as that derived in kinematics for motion down an inclined plane (see Part I., Arts. 164–166). The second equation gives the normal reaction of the line $N=mg\cos\theta$; hence, the pressure on the line, $-N$, is constant.

(*b*) If the line be rough, the second equation remains the same, while the first must be replaced by the following,

$$m\frac{dv}{dt}=mg\sin\theta-\mu N=mg(\sin\theta-\mu\cos\theta).$$

As the acceleration is constant whether there be friction or not, the motion is uniformly accelerated, unless $\sin\theta-\mu\cos\theta=0$, *i.e.* $\mu=\tan\theta$.

Find v and s; show that, in the exceptional case $\mu=\tan\theta$, the motion is uniform unless the initial velocity be zero; show that, for motion *up* the plane, the first equation becomes $dv/dt=-g(\sin\theta+\mu\cos\theta)$, the motion being uniformly retarded until $t=v_0/g(\sin\theta+\mu\cos\theta)$ when the particle either begins to move down the line or remains at rest.

(3) A string of length l (ft.) carries at one end a mass of m lbs. while the other end is fixed at a point O on a smooth horizontal table. The mass m is made to describe a circle of radius l about O on the table, with constant velocity $=v$ ft. per second. Show that the tension of the string is $=mv^2/l$ poundals.

(4) In Ex. (3), let $m = 2$ lbs.; $l = 3$ ft.; find the tension in pounds: (*a*) when the mass makes one revolution per second; (*b*) when it makes 10 revolutions per second. (*c*) If the string cannot stand a tension of more than 300 pounds, what is the greatest allowable velocity?

(5) A locomotive weighing 32 tons moves in a curve of 800 ft. radius with a velocity of 30 miles an hour; find the horizontal pressure on the rails.

(6) To prevent the lateral pressure on the rails in a curve, the track is inclined inwards. Determine the required elevation e of the outer above the inner rail for a given velocity v and radius R if the gauge (*i.e.* the distance between the rails) is 4 ft. 8 in.

(7) A plummet is suspended from the roof of a railroad car; how much will it be deflected from the vertical when the train is running 45 miles an hour in a curve of 300 yards' radius?

(8) A body on the surface of the earth partakes of the earth's daily rotation on its axis. The constraint holding it in its circular path is due to the attractive force of the earth. Taking the earth's equatorial radius as 3963 miles, show that the centripetal acceleration of a particle at the equator is about $\frac{1}{9}$ ft. per second, or about $\frac{1}{290}$ of the actually observed acceleration $g = 32 \cdot 09$ of a body falling *in vacuo*.

(9) If the earth were at rest, what would be the acceleration of a body falling *in vacuo* at the equator?

(10) Show that if the velocity of the earth's rotation were over 17 times as large as it actually is, the force of gravity would not be sufficient to detain a body near the surface at the equator.

(11) Show that in latitude ϕ the acceleration of a falling body, if the earth were at rest, would be $g_1 = g + j \cos^2 \phi$, where g is the observed acceleration of a falling body on the rotating earth and j the centripetal acceleration at the equator. Thus, in latitude $\phi = 45°$, $g = 980 \cdot 6$ cm.; hence $g_1 = 982 \cdot 3$.

(12) A chandelier weighing 75 lbs. is suspended from the ceiling of a hall by means of a chain $12\frac{1}{2}$ ft. long whose weight is neglected. By how much is the tension of the chain increased if it be set swinging so that the velocity at the lowest point is 5 ft. per second?

(13) A cord of 2 ft. length passes at its middle point through a hole in a smooth horizontal table. It carries at its lower end a mass of 2 lbs., at its other end a mass of 1 lb. The latter is set to revolve in a

circle about the hole so as to stretch the cord and just prevent the mass of 2 lbs. from descending. (*a*) How many revolutions must it make? (*b*) If only one-fourth of the cord lie on the table while three-fourths hang down, how many revolutions must be made?

(14) Show that, when a particle moves with constant velocity in a vertical circle, the constraining force F' (Art. 167) is always directed towards a fixed point on the vertical diameter.

175. *A particle of mass* m *subject to gravity alone is constrained to move in a vertical circle of radius* l. If there be no friction on the curve and the constraint be produced by a weightless rod or string joining the particle to the centre of the circle, we have the problem of the **simple mathematical pendulum.**

Equation (1), Art. 169, is readily seen to reduce in this case (see Fig. 25) to the form

$$l\frac{d^2\theta}{dt^2}+g\sin\theta=0. \tag{8}$$

A first integration gives, as shown in kinematics (Part I., Arts. 215, 216),

$$\tfrac{1}{2}v^2=g\left(l\cos\theta+\frac{v_0^{\,2}}{2g}-l\cos\theta_0\right), \tag{9}$$

where v_0 is the velocity which the particle has at the time $t=0$ when its radius makes the angle $AOP_0=\theta_0$ with the vertical. Multiplying by m, we have, for the kinetic energy of the particle,

$$\tfrac{1}{2}mv^2=mg(l\cos\theta+h), \tag{10}$$

where $h=v_0^{\,2}/2g-l\cos\theta_0$ is a constant. If the horizontal line MN, drawn at the height $v_0^{\,2}/2g$ above the initial point P_0, intersect the vertical diameter AB at R, it appears from the figure that $h=RO$.

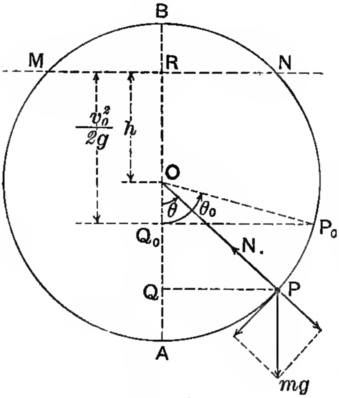

Fig. 25.

176. Taking R as origin and the axis of z vertically downwards, we have $RQ=z=l\cos\theta+h$; hence the force-function U has the simple expression

$$U=mgz;$$

and the velocity $v=\sqrt{2gz}$ is seen to become zero when the particle reaches the horizontal line MN.

For the further treatment of the problem, three cases must be distinguished according as this line of zero-velocity MN intersects the circle, touches it, or does not meet it at all; *i.e.* according as

$$h \lesseqgtr l, \text{ or } \frac{v^2}{2g} \lesseqgtr 2l\sin^2\frac{\theta}{2}. \tag{11}$$

177. Equation (2), Art. 169, serves to determine the reaction N of the circle, or the pressure $-N$ on the circle. We have

$$m\frac{v^2}{l}=-mg\cos\theta+N,$$

whence

$$N=m\left(\frac{v^2}{l}+g\cos\theta\right).$$

Substituting for v^2 its value from (10), we find

$$N=mg\left(2\frac{h}{l}+3\cos\theta\right). \tag{12}$$

The pressure on the curve has therefore its greatest value when $\theta=0$, *i.e.* at the lowest point A. It becomes zero for $l\cos\theta_1=-\frac{2}{3}h$, which is easily constructed.

178. If the constraint be complete as for a bead sliding along a circular wire, or a small ball moving within a tube, the pressure merely changes sign at the point $\theta=\theta_1$. But if the constraint be one-sided, the particle may at this point leave the circle. The one-sided constraint may be such that $OP\leqq l$, as when the particle runs in a groove cut on the inside of a ring, or when it is joined to the centre by a string; in this case the particle may leave the circle at some point of its upper half. Again, the one-sided constraint may be such that $OP \overset{=}{>} l$, as when the particle runs in a groove cut on the rim of a disc; in this case the particle can of course only move on the upper half of the circle.

179. Exercises.

(1) For $h = l$, equation (10) can be integrated in finite terms. Show that in this limiting case the particle approaches the highest point B of the circle asymptotically, reaching it only in an infinite time.

(2) Derive the equations of motion for the problem of the simple pendulum (Art. 175) from the general equations of Arts. 172, 173.

(3) For $\theta_0 = 60°$, $l = 1$ ft., $v_0 = 9$ ft. per second, show that the particle will leave the circle very nearly at the point $\theta_1 = 120°$, if the constraint be such that $OP \leqq l$ (Art. 178).

(4) For $v_0 = 10$ ft. per second, everything else being as in Ex. (3), show that the particle will leave the circle at the point $\theta_1 = 134\frac{1}{2}°$, nearly.

(5) A particle, subject to gravity and constrained to the inside of a vertical circle ($OP \leqq l$), makes complete revolutions. Show that it cannot leave the circle at any point, if $\frac{2}{3}h > l$; and that it will leave the circle at the point for which $\cos\theta = -\frac{2}{3}h/l$, if $\frac{2}{3}h < l$.

(6) In the experiment of swinging in a vertical circle a glass containing water, and suspended by means of a string, if the string be 2 ft. long, what must be the velocity at the lowest point if the experiment is to succeed?

(7) A particle subject to gravity moves on the outside of a vertical circle; determine where it will leave the circle: (*a*) if MN (Fig. 25) intersects the circle; (*b*) if MN touches the circle; (*c*) if MN does not meet the circle.

(8) A particle subject to gravity is compelled to move on any vertical curve $z = f(x)$ without friction. Show that the velocity at any point is $v = \sqrt{2gz}$ (comp. Art. 176) if the horizontal axis of x be taken at a height above the initial point equal to the "height due to the initial velocity," *i.e.* $v_0^2/2g$.

(9) Investigate the motion of a particle subject to gravity, and compelled to move on a circle whose plane is inclined to the horizon at an angle α.

(10) A particle constrained to a straight line is attracted to a fixed centre outside this line, the attraction being proportional to the distance from the centre. Determine its motion.

180. Motion on Any Fixed Curve without Friction. The position of a point on a curve can always be determined by a single variable. Thus, for instance, the length s of the curve counted from some origin on the curve might be taken as this variable; if the curve be a circle, the polar angle θ might be selected; on an ellipse, the eccentric angle ϕ; on a cycloid, the angle through which the generating circle has rolled, etc. We shall designate this variable by q, and write the equations of the curve in the form

$$x=f_1(q), \quad y=f_2(q), \quad z=f_3(q). \tag{13}$$

The expression for the velocity v is in this case

$$v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=\left[\left(\frac{df_1}{dq}\right)^2+\left(\frac{df_2}{dq}\right)^2+\left(\frac{df_3}{dq}\right)^2\right]\left(\frac{dq}{dt}\right)^2. \tag{14}$$

If there is no friction, the real equation of motion is the equation (1) of Art. 169, which is equivalent to the equation of kinetic energy (7), Art. 173; when the variable q is introduced, this equation becomes

$$d(\tfrac{1}{2}mv^2)=\left(X\frac{df_1}{dq}+Y\frac{df_2}{dq}+Z\frac{df_3}{dq}\right)dq, \tag{15}$$

where v^2 is given by (14).

Putting, for shortness,

$$X\frac{df_1}{dq}+Y\frac{df_2}{dq}+Z\frac{df_3}{dq}=Q, \tag{16}$$

we can write the equation of motion in the simple form

$$d(\tfrac{1}{2}mv^2)=Qdq. \tag{17}$$

181. In the most general case, the given forces X, Y, Z will depend not only on the position of the particle, but also on its velocity and on the time. In this case, Q would be a function of q, dq/dt, and t; and equation (17) represents a differential equation of the second order between q and t.

If, however, the resultant F of the given forces depends only on the position of the particle so that Q is a function of q alone, the right-hand member of (17) is an exact differential, and a first integration can at once be performed. Then, substituting for v^2 its value from (14) in terms of q and dq/dt, we find a differential equation of the first order whose integration gives t in function of q.

182. Exercise.

A particle of mass m is constrained to a common helix $x = a\cos\theta$, $y = a\sin\theta$, $z = \kappa\theta$, whose axis is vertical. The particle is subject to gravity and is attracted by a centre situated on the axis, with a force directly proportional to the distance. Determine the motion.

3. MOTION ON A FIXED SURFACE.

183. Just as for motion on a curve (Art. 172), we find the general equations of motion

$$m\frac{d^2x}{dt^2} = X + N_x - \mu N\frac{dx}{ds},$$

$$m\frac{d^2y}{dt^2} = Y + N_y - \mu N\frac{dy}{ds}, \qquad (1)$$

$$m\frac{d^2z}{dt^2} = Z + N_z - \mu N\frac{dz}{ds}.$$

The normal reaction

$$N = \sqrt{N_x^2 + N_y^2 + N_z^2} \qquad (2)$$

being at right angles to the constraining surface

$$\phi(x, y, z) = 0, \qquad (3)$$

the following condition must be satisfied:

$$\frac{N_x}{\phi_x}=\frac{N_y}{\phi_y}=\frac{N_z}{\phi_z}, \tag{4}$$

where ϕ_x, ϕ_y, ϕ_z denote, as usual, the partial derivatives of $\phi(x, y, z)$ with regard to x, y, z, respectively.[1]

If the acceleration of the particle be zero, the equations (1) reduce to the conditions of equilibrium of a particle on a surface, as given in Statics (Part II., Art. 222).

184. *A particle of mass* m, *subject to gravity, is constrained to remain on the surface of a sphere of radius* r. If the constraint is produced by a weightless rod or string joining the particle to the centre of the sphere, the rod or string describes a cone, and the apparatus is called a **conical** or **spherical pendulum.**

Taking the centre O of the sphere as origin (Fig. 26), and the axis of z vertically downwards, we have for the equation of the sphere

$$x^2+y^2+z^2-r^2=0, \tag{5}$$

whence $\phi_x/x=\phi_y/y=\phi_z/z$. The direction cosines of N are $-x/r$, $-y/r$, $-z/r$. Hence, the equations of motion, as there is no friction:

$$m\ddot{x}=-N\frac{x}{r},\quad m\ddot{y}=-N\frac{y}{r},\quad m\ddot{z}=mg-N\frac{z}{r}. \tag{6}$$

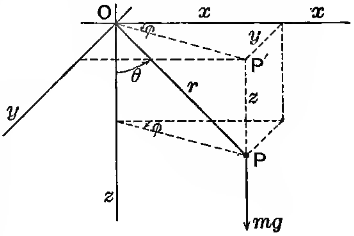

Fig. 26.

---

[1] This abridged notation is readily extended to the second and higher derivatives: $\phi_{xx}=\frac{\partial^2\phi}{\partial x^2}$, $\phi_{xy}=\frac{\partial^2\phi}{\partial x\partial y}$, etc. It will also sometimes be convenient to use the fluxional notation for derivatives with respect to the time

$$\frac{dx}{dt}=\dot{x},\quad \frac{dy}{dt}=\dot{y},\quad \frac{dz}{dt}=\dot{z};\quad \frac{d^2x}{dt^2}=\ddot{x},\quad \frac{d^2y}{dt^2}=\ddot{y},\quad \frac{d^2z}{dt^2}=\ddot{z};$$

$$\frac{dr}{dt}=\dot{r},\quad \frac{d\theta}{dt}=\dot{\theta};\quad \frac{d^2r}{dt^2}=\ddot{r},\text{ etc.}$$

In mechanics, this notation is of particular advantage, not only because the time so often appears as the independent variable, but also because the initial values of these derivatives (*i.e.* the components of the initial velocity and acceleration) can then be indicated by zero subscripts. Thus, the components of the initial velocity would be $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$.

As the resistance N does no work, the principle of kinetic energy gives

$$\tfrac{1}{2} mv^2 = mgz + C,$$

or, dividing by $\tfrac{1}{2} m$,

$$v^2 = 2(gz + h). \tag{7}$$

To determine the constant of integration h, we have $v = v_0$ when $z = z_0$; hence

$$v^2 = v_0^2 + 2g(z - z_0). \tag{8}$$

185. Another first integral is found by applying the principle of areas which holds for the projection of the motion on the horizontal xy-plane. This appears by considering that N is always directed along the radius of the sphere so that the resultant of N and the weight mg of the particle always intersects the axis of z (see Art. 93). We have therefore

$$x\dot{y} - y\dot{x} = c, \tag{9}$$

where $\tfrac{1}{2} c$ is the sectorial velocity of the projection OP' of the radius $OP = r$ on the xy-plane.

186. For the further treatment of the problem it is best to introduce polar co-ordinates (Fig. 26). Let θ be the angle between r and the axis of z, ϕ that between the projection OP' of r on the xy-plane and the axis of x; then

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta;$$

and

$$\begin{aligned} \dot{x} &= r \cos\theta \cos\phi \cdot \dot{\theta} - r \sin\theta \sin\phi \cdot \dot{\phi}, \\ \dot{y} &= r \cos\theta \sin\phi \cdot \dot{\theta} + r \sin\theta \cos\phi \cdot \dot{\phi}, \\ \dot{z} &= - r \sin\theta \cdot \dot{\theta}. \end{aligned}$$

Hence

$$\begin{aligned} v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= r^2(\dot{\theta}^2 + \sin^2\theta \cdot \dot{\phi}^2), \\ x\dot{y} - y\dot{x} &= r^2 \sin^2\theta \cdot \dot{\phi}. \end{aligned}$$

The first integrals (7) and (9) thus become in polar co-ordinates

$$r^2(\dot{\theta}^2 + \sin^2\theta \cdot \dot{\phi}^2) = 2(gr\cos\theta + h), \tag{10}$$

$$r^2 \sin^2\theta \cdot \dot{\phi} = c. \tag{11}$$

187. The constants of integration h, c can now be determined if the initial values of θ, $\dot{\theta}$, $\dot{\phi}$ are given.

Eliminating dt between the equations (10) and (11), we find the differential equation of the path

$$d\phi = \frac{c\,d\theta}{\sin\theta\sqrt{2r^2\sin^2\theta(gr\cos\theta + h) - c^2}}, \tag{12}$$

whose integration gives the equation of the path in the spherical coordinates θ, ϕ (colatitude and longitude).

On the other hand, if $\dot{\phi}$ be eliminated between (10) and (11), we find the relation

$$dt = \frac{r^2\sin\theta\,d\theta}{\sqrt{2r^2\sin^2\theta(gr\cos\theta + h) - c^2}}, \tag{13}$$

which, upon integration, determines the time as a function of θ, or the position of the point in its path at any given time.

The equations (12) and (13) contain therefore the complete solution of our problem, with the exception of the determination of the resistance N. Their discussion cannot here be given, as they lead to elliptic integrals.

188. To find the resistance N, multiply the equations (6) by x, y, z and add; this gives

$$m(x\ddot{x} + y\ddot{y} + z\ddot{z}) = mgz - Nr. \tag{14}$$

Differentiating twice the equation of the sphere (5), we find

$$x\ddot{x} + y\ddot{y} + z\ddot{z} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 0,$$

or since $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$,

$$x\ddot{x} + y\ddot{y} + z\ddot{z} = -v^2.$$

Substituting this value in (14), we find

$$N = \frac{m(gz + v^2)}{r}$$

or, by (8),
$$N = \frac{m}{r}(3gz + v_0^2 - 2gz_0). \tag{15}$$

If the constraint be one-sided as in the case of a string pendulum, the particle will leave the sphere whenever in its upper half z becomes $\overset{=}{>} \frac{2}{3}z_0 - v_0^2/3g$.

189. That particular case of the problem of the conical pendulum in which the particle moves in a horizontal circle can be treated directly in an elementary manner. It finds its application in the theory of the *governor of a steam engine.*

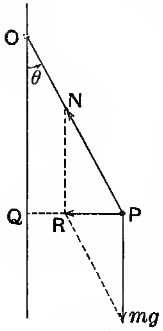

Fig. 27.

Let O (Fig. 27) be the point of suspension, $OP = l$ the length of the pendulum rod, $\measuredangle\, QOP = \theta$ the constant angle of inclination of the rod to the vertical OQ. The only forces acting on the particle are its weight mg and the tension of the rod. As both these forces lie in the normal plane of the path, the tangential acceleration is zero, and the particle moves uniformly in the circle.

The radius of curvature of the path is the radius $QP = l \sin\theta$ of the horizontal circle. The resultant R of mg and N must act along the radius; its magnitude is seen from the figure to be $R = mg \tan\theta$. Hence the equation (2) of Art. 169 gives

$$m \frac{v^2}{l \sin\theta} = mg \tan\theta,$$

or,

$$v^2 = gl \frac{\sin^2\theta}{\cos\theta}. \tag{16}$$

The figure also shows that the tension of the rod is

$$N = \frac{mg}{\cos\theta}. \tag{17}$$

190. Exercises.

(1) Show that the time of revolution T of the conical pendulum (Art. 186) is the same as the time of one complete oscillation of a simple pendulum of length $l \cos\theta$.

(2) Show that the angular velocity with which the vertical plane of the rod turns about the vertical axis OQ (Fig. 27) is inversely proportional to the cosine of the angle θ.

(3) A conical pendulum makes $n = 60$ revolutions per minute: (*a*) What is the height of the cone? (*b*) If the mass of the bob be $m = 1$ oz., and the length of the rod $l = 1$ ft., what is the tension of the rod? ($g = 32{\cdot}2$.)

(4) From the equations (5) and (6), Art. 184, derive the approximate path of the bob of a simple pendulum when the oscillations are very small.

191. Motion on Any Fixed Surface without Friction. The position of a point on a surface can always be determined by two variables, say q_1, q_2. Thus, on a sphere, the latitude and longitude of a point determine its position; and on any surface the two systems of curves known as the curves of curvature of the surface might serve as a system of co-ordinates. In other words, the motion of a point on a surface is really a problem of motion in two dimensions, just as the motion on a curve takes place in one dimension (Art. 180).

Let $$x=f_1(q_1, q_2),\ y=f_2(q_1, q_2),\ z=f_3(q_1, q_2) \tag{18}$$

be the equations of the given surface, so that the elimination of q_1, q_2 from these equations would give the ordinary equation $\phi(x, y, z)=0$ of the surface. Then we have for the velocity v the expression

$$\begin{aligned} v^2 &= \dot{x}^2+\dot{y}^2+\dot{z}^2 \\ &= \left(\frac{\partial f_1}{\partial q_1}\dot{q}_1+\frac{\partial f_1}{\partial q_2}\dot{q}_2\right)^2+\left(\frac{\partial f_2}{\partial q_1}\dot{q}_1+\frac{\partial f_2}{\partial q_2}\dot{q}_2\right)^2+\left(\frac{\partial f_3}{\partial q_1}\dot{q}_1+\frac{\partial f_3}{\partial q_2}\dot{q}_2\right)^2. \end{aligned} \tag{19}$$

If there be no friction, the equation of kinetic energy gives

$$d(\tfrac{1}{2}mv^2)=$$

$$\left(X\frac{\partial f_1}{\partial q_1}+Y\frac{\partial f_2}{dq_1}+Z\frac{\partial f_3}{\partial q_1}\right)dq_1+\left(X\frac{\partial f_1}{\partial q_2}+Y\frac{\partial f_2}{\partial q_2}+Z\frac{\partial f_3}{\partial q_2}\right)dq_2,$$

or say

$$d(\tfrac{1}{2}mv^2)=Q_1dq_1+Q_2dq_2. \tag{20}$$

If the forces depend only on the position of the particle, Q_1 and Q_2 are functions of q_1, q_2 alone; if, moreover, the expression $Q_1dq_1+Q_2dq_2$ is an exact differential of a function of q_1 and q_2, say $dU(q_1, q_2)$, the equation (20) gives at once a first integral

$$\tfrac{1}{2}mv^2=U(q_1, q_2)+h. \tag{21}$$

4. MOTION ON A MOVING OR VARIABLE CURVE OR SURFACE.

192. If the constraining curve or surface be not fixed and invariable, the conditional equations will contain the time t explicitly, besides the co-ordinates x, y, z of the moving particle (Art. 164). For the sake of simplicity we here assume the curve or surface to be smooth, so as to offer only a *normal* resistance N; if there be friction, the components of the frictional resistance may be regarded as included in the components X, Y, Z of the resultant force acting on the particle.

The treatment of this general problem of constrained motion of a particle is here presented not so much on account of its application to the solution of particular problems, as for the reason that it offers an opportunity of explaining the meaning of d'Alembert's principle and illustrating its application in a comparatively simple case.

193. Two Constraints. Let the equations of the curve to which the particle is constrained be

$$\phi\,(x, y, z, t)=0, \quad \psi\,(x, y, z, t)=0. \tag{1}$$

To apply d'Alembert's principle (Arts. 97–102), let the particle be subjected, at any given time t, to an infinitesimal displacement δs. If this displacement be selected along the curve (1), the reaction N of the curve, being at right angles to δs, will do no work during the displacement; hence the equation of motion will be the same as that for a free particle (see Art. 101), viz.

$$(-m\ddot{x}+X)\delta x+(-m\ddot{y}+Y)\delta y+(-m\ddot{z}+Z)\delta z=0. \tag{2}$$

In this equation, then, the forces X, Y, Z do not involve the normal reaction of the curve; but the components δx, δy, δz of the displacement δs must be selected so that δs should lie on the curve (1) at the time t; this is usually expressed by saying that *the displacement should be compatible with the conditions* (1).

Some authors confine the term *virtual displacement* to displacements compatible with the given conditions.

The displacement $\delta s = \sqrt{\delta x^2 + \delta y^2 + \delta z^2}$ will be compatible with the conditions (1) at the given time t if the following conditions, obtained by differentiating the equations (1), are satisfied:

$$\begin{aligned} \phi_x \delta x + \phi_y \delta y + \phi_z \delta z &= 0, \\ \psi_x \delta x + \psi_y \delta y + \psi_z \delta z &= 0. \end{aligned} \tag{3}$$

It should be noticed that in this differentiation the time t is regarded as constant, the displacement being taken to occur *at a given time.*

The equations (2) and (3), which must be fulfilled simultaneously, constitute the equations of motion of our problem.

By means of the equations (3), two of the component displacements δx, δy, δz can be eliminated from the equation (2); the coefficient of the third equated to zero gives the actual equation of motion.

194. To perform this elimination systematically *the method of indeterminate multipliers* can be used as follows. Multiplying the equations (3) respectively by the indeterminate factors λ and μ and adding them to the equation (2), we obtain the single equation

$$\begin{aligned} (-m\ddot{x} + X + \lambda\phi_x + \mu\psi_x)\delta x + (-m\ddot{y} + Y + \lambda\phi_y + \mu\psi_y)\delta y \\ + (-m\ddot{z} + Z + \lambda\phi_z + \mu\psi_z)\delta z = 0, \end{aligned}$$

in which the arbitrary quantities λ, μ can be so selected as to make the coefficients of two of the three displacements δx, δy, δz vanish; the coefficient of the third must then also vanish. The equation is therefore equivalent to the following three equations:

$$\begin{aligned} m\ddot{x} &= X + \lambda\phi_x + \mu\psi_x, \\ m\ddot{y} &= Y + \lambda\phi_y + \mu\psi_y, \\ m\ddot{z} &= Z + \lambda\phi_z + \mu\psi_z. \end{aligned} \tag{4}$$

These equations (4), in connection with the two conditions (1), are sufficient to determine the five quantities x, y, z, λ, μ as functions of the time; the values of x, y, z so found give the position of the particle at any time, while λ, μ can be shown to determine the pressure on the curve.

195. To find the reaction N of the curve, let us compare the equations (4) with the equations of Art. 161. It appears at once that the forces X', Y', Z' that would replace the conditions (1), *i.e.* the components of the reaction N of the constraining curve, are

$$X' = \lambda\phi_x + \mu\psi_x, \quad Y' = \lambda\phi_y + \mu\psi_y, \quad Z' = \lambda\phi_z + \mu\psi_z,$$

whence

$$N^2 = (\lambda\phi_x + \mu\psi_x)^2 + (\lambda\phi_y + \mu\psi_y)^2 + (\lambda\phi_z + \mu\psi_z)^2. \tag{5}$$

These equations determine the magnitude and direction of the reaction N, as soon as λ and μ are found.

196. Let us now combine the equations (4) according to the principle of kinetic energy; that is, multiply them by dx, dy, dz, and add. The left-hand member becomes, of course, the exact differential $d(\frac{1}{2}mv^2)$. The right-hand member,

$$Xdx + Ydy + Zdz + \lambda(\phi\, dx + \phi_y dy + \phi_z dz) + \mu(\psi_x dx + \psi_y dy + \psi_z dz),$$

will in general contain terms depending on the reaction of the surface; in other words, in the *actual* displacement $ds = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$ of the particle the reaction of the moving curve will in general do work.

197. Only in the particular case when the curve is fixed will the work of the reaction be zero; for in this case the conditional equations (1) do not contain the time explicitly, and their complete differentiation gives the relations

$$\phi_x dx + \phi_y dy + \phi_z dz = 0, \quad \psi_x dx + \psi_y dy + \psi_z dz = 0,$$

which show that the coefficients of λ and μ in the equation of kinetic energy vanish.

Hence, *for motion on a fixed curve* we have

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz, \tag{6}$$

which agrees with the equation (7) of Art. 173, considering that the frictional resistance is supposed to be included among the forces X, Y, Z.

198. In the general case, the complete differentiation of the equations (1) gives

$$\phi_x dx + \phi_y dy + \phi_z dz + \phi_t dt = 0,$$
$$\psi_x dx + \psi_y dy + \psi_z dz + \psi_t dt = 0;$$

and the equation of kinetic energy *for motion on a moving or variable curve* becomes

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz - (\lambda\phi_t + \mu\psi_t)dt. \tag{7}$$

The distinction between the *virtual* displacement δs along the curve in its position at the time t and the *actual* displacement ds of the particle along the moving curve should be clearly understood. The virtual displacement $\delta s = PP'$ joins the position P (x, y, z) of the particle at the time t to a point P' $(x+\delta x, y+\delta y, z+\delta z)$, which is on the curve, and infinitely near to P at the time t, while the actual displacement $ds = PP''$ joins P (x, y, z) to the position P'' $(x+dx, y+dy, z+dz)$ of the particle at the time $t+dt$; P'' lies, therefore, on the position that the curve has, not at the time t, but at the time $t+dt$. The reaction N of the curve at the time t is normal to δs, but not to ds.

199. One Constraint. Let

$$\phi(x, y, z, t) = 0 \tag{8}$$

be the equation of the surface on which the particle is assumed to remain throughout its motion. The reaction N of this surface will do no work if the displacement δs be taken along the position of the surface at a given time t. In other words, to obtain a displacement δs compatible with the condition (8), its components δx, δy, δz should satisfy the condition

$$\phi_x \delta x + \phi_y \delta y + \phi_z \delta z = 0, \tag{9}$$

obtained by differentiating the equation (8) with respect to the co-ordinates.

200. By means of the relation (9), one of the displacements δx, δy, δz can be eliminated from the general equation of motion (2); the two remaining displacements will then be independent, and their coefficients can therefore be equated to zero separately.

The elimination is again conveniently effected by the method of indeterminate multipliers. Multiplying equation (9) by an indeterminate factor λ, and adding it to equation (2), we find the single equation

$$\begin{aligned} &(-m\ddot{x} + X + \lambda\phi_x)\, \delta x \\ &+ (-m\ddot{y} + Y + \lambda\phi_y)\, \delta y \\ &+ (-m\ddot{z} + Z + \lambda\phi_z)\, \delta z = 0, \end{aligned}$$

in which the arbitrary quantity λ can be so selected as to make the coefficient of one of the three displacements vanish. The other two displacements being arbitrary, their coefficients must also vanish. The last equation can therefore be replaced by the following three:

$$m\ddot{x} = X + \lambda\phi_x, \quad m\ddot{y} = Y + \lambda\phi_y, \quad m\ddot{z} = Z + \lambda\phi_z, \tag{10}$$

which, in connection with the given condition (8), fully determine the problem; for they are sufficient for finding x, y, z, and λ as functions of t.

201. Just as in Art. 195, it follows that the components of the reaction N of the surface are

$$X' = \lambda\phi_x, \quad Y' = \lambda\phi_y, \quad Z' = \lambda\phi_z,$$

whence

$$N = \lambda\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}. \tag{11}$$

202. If the equations (10) be combined according to the principle of kinetic energy, we find

$$d(\tfrac{1}{2}mv^2)=Xdx+Ydy+Zdz+\lambda(\phi_x dx+\phi_y dy+\phi_z dz),$$

where again the coefficient of λ vanishes only when the surface is fixed, in which case

$$d(\tfrac{1}{2}mv^2)=Xdx+Ydy+Zdz; \qquad (12)$$

while in the general case of a moving or variable surface we have

$$d(\tfrac{1}{2}mv^2)=Xdx+Ydy+Zdz-\lambda\phi_t dt. \qquad (13)$$

203. Plane Motion. If a particle be constrained to move in a plane curve under the action of forces lying in the plane of the curve, d'Alembert's principle gives the equation of motion

$$(-m\ddot{x}+X)\delta x+(-m\ddot{y}+Y)\delta y=0; \qquad (14)$$

and the equation of the curve

$$\phi(x, y, t)=0 \qquad (15)$$

gives by differentiation for a virtual displacement δs on the curve at a given time t,

$$\phi_x\delta x+\phi_y\delta y=0. \qquad (16)$$

Hence, proceeding as in Art. 200, the equations of motion can be written in the form

$$m\ddot{x}=X+\lambda\phi_x, \quad m\ddot{y}=Y+\lambda\phi_y, \qquad (17)$$

while the normal reaction of the curve is

$$N=\lambda\sqrt{\phi_x^2+\phi_y^2}. \qquad (18)$$

204. The process of solution is now as follows for the case of plane motion. Differentiate the equation of condition (15), which holds at any time, with respect to the time, remembering that x and y are functions of the time; this gives:

$$\frac{d\phi}{dt}\equiv\dot{x}\phi_x+\dot{y}\phi_y+\phi_t=0. \qquad (19)$$

Differentiating again, we find

$$\frac{d^2\phi}{dt^2} \equiv \ddot{x}\phi_x + \ddot{y}\phi_y + \dot{x}^2\phi_{xx} + 2\dot{x}\dot{y}\phi_{xy} + \dot{y}^2\phi_{yy}$$
$$+ 2\dot{x}\phi_{tx} + 2\dot{y}\phi_{ty} + \phi_{tt} = 0. \qquad (20)$$

If in this last equation the values of $\ddot{x}$, $\ddot{y}$ be substituted from (17), we have a linear equation for λ. The value of λ thus obtained can then be introduced into the equations of motion (17); and it only remains to integrate these equations.

This integration will often be facilitated by introducing new variables for x, y.

205. *A particle moves without friction in a straight tube which revolves uniformly in a horizontal plane about one of its points. Determine its motion.*

To illustrate the application of the general methods, we shall solve this problem completely, first without the use of indeterminate multipliers, and then with their aid, although the problem is so simple that it might be solved without applying these general methods, as will be pointed out below.

As the weight of the particle is balanced by the vertical reaction of the tube, we have a case of plane motion with $X = 0$, $Y = 0$. Hence d'Alembert's equation (14) becomes

$$\ddot{x}\delta x + \ddot{y}\delta y = 0. \qquad (21)$$

If we take as origin the point O about which the tube rotates, the constraining curve is a straight line through the origin $y = x\tan\theta$, where $\theta = \omega t$, ω being the constant angular velocity of the tube and the axis of x coinciding with the initial position of the tube at the time $t = 0$. Hence

$$\begin{aligned} x &= r\cos\omega t, \quad y = r\sin\omega t; \\ \delta x &= \delta r\cos\omega t, \quad \delta y = \delta r\sin\omega t; \\ \dot{x} &= \dot{r}\cos\omega t - \omega r\sin\omega t, \quad \dot{y} = \dot{r}\sin\omega t + \omega r\cos\omega t; \end{aligned} \qquad (22)$$

$$\ddot{x} = \ddot{r}\cos\omega t - 2\omega\dot{r}\sin\omega t - \omega^2 r\cos\omega t, \quad \ddot{y} = \ddot{r}\sin\omega t + 2\omega\dot{r}\cos\omega t - \omega^2 r\sin\omega t.$$

Substituting these values of $\ddot{x}$, $\ddot{y}$ and δx, δy into the equation of motion (21), we find after reduction

$$\ddot{r} - \omega^2 r = 0. \qquad (23)$$

As mentioned above, this equation might have been derived directly by considering that the acceleration along the tube is due to the centrifugal force alone (see Art. 170).

206. The general integral of equation (23) is

$$r = c_1 e^{\omega t} + c_2 e^{-\omega t}.$$

If $r = r_0$ and $\dot{r} = v_0$ when $t = 0$, we have $r_0 = c_1 + c_2$, $v_0 = \omega(c_1 - c_2)$; hence

$$2\,\omega r = (\omega r_0 + v_0) e^{\omega t} + (\omega r_0 - v_0) e^{-\omega t}. \tag{24}$$

With $v_0 = 0$, $r_0 = 1$, this equation represents a common catenary. If $v_0 = r_0 \omega$, the equation reduces to the form $r = r_0 e^{\omega t}$, whence $t = (1/\omega) \log (r/r_0)$.

The minimum of r in (24) occurs for

$$t = \frac{1}{2\omega} \log \frac{\omega r_0 - v_0}{\omega r_0 + v_0};$$

its value is $r_1 = \sqrt{r_0^2 - (v_0/\omega)^2}$. It is easy to see that such a minimum can occur only when v_0 is negative and $> \omega r_0$ numerically.

207. To apply the method of indeterminate multipliers to our problem, let the equation of the tube be written in the form

$$\phi(x, y, t) \equiv x \cos \omega t - y \sin \omega t = 0. \tag{25}$$

Then we have $\phi_x = \cos \omega t$, $\phi_y = -\sin \omega t$, $\phi_t = -\omega(x \sin \omega t + y \cos \omega t)$; hence equation (16) assumes the form

$$\delta x \cos \omega t - \delta y \sin \omega t = 0;$$

and the equations of motion (17) are

$$m\ddot{x} = \lambda \cos \omega t, \quad m\ddot{y} = -\lambda \sin \omega t. \tag{26}$$

We have also $\phi_{xx} = 0$, $\phi_{xy} = \phi_{yx} = 0$, $\phi_{yy} = 0$, $\phi_{tx} = \phi_{xt} = -\omega \sin \omega t$, $\phi_{ty} = \phi_{yt} = -\omega \cos \omega t$, $\phi_{tt} = -\omega^2(x \cos \omega t - y \sin \omega t) = 0$. Hence, by (20),

$$\frac{d^2\phi}{dt^2} \equiv \ddot{x} \cos \omega t - \ddot{y} \sin \omega t - 2\,\omega \dot{x} \sin \omega t - 2\,\omega \dot{y} \cos \omega t = 0.$$

Substituting in this equation the values of $\ddot{x}, \ddot{y}$ from (26), we find the linear equation for λ which gives

$$\frac{\lambda}{m} = 2\,\omega(\dot{x} \sin \omega t + \dot{y} \cos \omega t) \sec 2\,\omega t.$$

Introducing this value into the equations (26), we have the differential equations of our problem in the form

$$\ddot{x} = 2\omega(\dot{x} \sin \omega t + \dot{y} \cos \omega t) \frac{\cos \omega t}{\cos 2\omega t}, \quad y = -2\omega(\dot{x} \sin \omega t + \dot{y} \cos \omega t) \frac{\sin \omega t}{\cos 2\omega t}.$$

Their integration can best be performed by introducing the radius vector r by means of the relations (22). Multiplying the equations respectively by $\cos\omega t$ and $\sin\omega t$, and adding, we find that the right-hand member vanishes, and we have

$$\ddot{x}\cos\omega t + \ddot{y}\sin\omega t = 0,$$

or, substituting for $\ddot{x}$ and y their values from (22),

$$\ddot{r} - \omega^2 r = 0,$$

which agrees with (23), Art. 205.

208. For the pressure on the curve, we have, by (18), since $\phi_x^2 + \phi_y^2 = 1$,

$$N = \lambda = \frac{2\,m\omega}{\cos 2\,\omega t}(\dot{x}\sin\omega t + \dot{y}\cos\omega t).$$

Substituting from (22) and (24), and reducing, we find

$$N = m\omega[(\omega r_0 + v_0)e^{\omega t}(1 + \tan 2\,\omega t) + (\omega r_0 - v_0)e^{-\omega t}(1 - \tan 2\,\omega t)].$$

209. Exercises.*

(1) A particle subject to gravity moves without friction in a straight tube which revolves uniformly in a vertical circle. Find the distance r of the particle from the centre of rotation at any time t.

(2) A particle moves without friction in a circular tube which rotates uniformly in a horizontal plane about a point O in its circumference. If the particle is at the time $t = 0$ at rest at the end of the diameter passing through O, what is its position at any time t?

(3) A particle moves in a horizontal circular tube whose radius increases proportionally to the time. At the time $t = 0$ the radius is a, and the particle has a velocity v_0 perpendicular to the radius. Find the position and velocity at any time t.

---

* These examples are taken from Walton's Collection (referred to in Art. 159), pp. 401–406.

V. *Lagrange's Form of the Equations of Motion.*

210. It has been shown in Arts. 180, 181, how the equations of motion on a fixed curve can be made to depend on a single variable q, and in Art. 191 how the motion on a fixed surface can be expressed by means of two variables q_1, q_2. By applying this idea, and by introducing the kinetic energy T and its derivatives, the equations of motion of a particle with or without conditions can be put into a remarkably compact form, which was first devised by Lagrange for the general equations of motion of a system of n particles (comp. Arts. 385–394). We proceed to establish these equations, first for the case of motion on a variable curve, then for motion on a variable surface, and finally for a free particle.

211. Particle Subject to Two Conditions. As shown in Art. 194 (comp. Art. 192), the equations of motion in Cartesian co-ordinates can be written in the form

$$\begin{aligned} m\ddot{x} &= X + \lambda\phi_x + \mu\psi_x, \\ m\ddot{y} &= Y + \lambda\phi_y + \mu\psi_y, \\ m\ddot{z} &= Z + \lambda\phi_z + \mu\psi_z, \end{aligned} \tag{1}$$

if the equations of condition are

$$\phi(x, y, z, t) = 0, \qquad \psi(x, y, z, t) = 0. \tag{2}$$

The single variable q, that determines the position of the particle on the curve, is called the *Lagrangian*, or *generalized, co-ordinate* of the particle. The Cartesian co-ordinates, x, y, z, are functions of the Lagrangian co-ordinate q, and of the time t, say

$$x = f_1(t, q), \quad y = f_2(t, q), \quad z = f_3(t, q). \tag{3}$$

To introduce q in the place of x, y, z, we shall need the derivatives $\dot{x}$, $\dot{y}$, $\dot{z}$. The first of the equations (3) gives

$$\dot{x} = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial q}\dot{q},$$

so that $\dot{x}$ can be regarded as a function of t, q, and $\dot{q}=dq/dt$. Hence we have

$$\frac{\partial \dot{x}}{\partial q}=\frac{\partial^2 f_1}{\partial q \partial t}+\frac{\partial^2 f_1}{\partial q^2}\dot{q}, \quad \frac{\partial \dot{x}}{\partial \dot{q}}=\frac{\partial f_1}{\partial q}.$$

In the former of these expressions, the right-hand member can be put into the more compact form $\frac{d}{dt}\frac{\partial f_1}{\partial q}$, as is easily verified by carrying out the indicated differentiation with respect to t. As similar results hold for y and z, we have

$$\frac{\partial \dot{x}}{\partial q}=\frac{d}{dt}\frac{\partial f_1}{\partial q}, \quad \frac{\partial \dot{y}}{\partial q}=\frac{d}{dt}\frac{\partial f_2}{\partial q}, \quad \frac{\partial \dot{z}}{\partial q}=\frac{d}{dt}\frac{\partial f_3}{\partial q}, \tag{4}$$

$$\frac{\partial \dot{x}}{\partial \dot{q}}=\frac{\partial f_1}{\partial q}, \quad \frac{\partial \dot{y}}{\partial \dot{q}}=\frac{\partial f_2}{\partial q}, \quad \frac{\partial \dot{z}}{\partial \dot{q}}=\frac{\partial f_3}{\partial q}. \tag{5}$$

212. Let us now add the equations of motion (1) after multiplying them by $\partial f_1/\partial q$, $\partial f_2/\partial q$, $\partial f_3/\partial q$. The coefficient of λ in the resulting equation, viz.

$$\frac{\partial \phi}{\partial x}\frac{\partial f_1}{\partial q}+\frac{\partial \phi}{\partial y}\frac{\partial f_2}{\partial q}+\frac{\partial \phi}{\partial z}\frac{\partial f_3}{\partial q},$$

is equal to zero, since it is evidently proportional to the cosine of the angle made at a given time by the tangent to the curve (3) with the normal to the surface $\phi=0$. For a similar reason, the coefficient of μ vanishes; and the resulting equation is

$$m\left(\ddot{x}\frac{\partial f_1}{\partial q}+\ddot{y}\frac{\partial f_2}{\partial q}+\ddot{z}\frac{\partial f_3}{\partial q}\right)=Q, \tag{6}$$

if, as in Arts. 180, 181, we put for shortness

$$X\frac{\partial f_1}{\partial q}+Y\frac{\partial f_2}{\partial q}+Z\frac{\partial f_3}{\partial q}=Q.$$

This quantity Q can evidently be expressed as a function of t, q, and $\dot{q}$.

The equation (6) can also be written in the form

$$m\frac{d}{dt}\left(\dot{x}\frac{\partial f_1}{\partial q}+\dot{y}\frac{\partial f_2}{\partial q}+\dot{z}\frac{\partial f_3}{\partial q}\right)-m\left(\dot{x}\frac{d}{td}\frac{\partial f_1}{\partial q}+\dot{y}\frac{d}{dt}\frac{\partial f_2}{\partial q}+\dot{z}\frac{d}{dt}\frac{\partial f_3}{\partial q}\right)=Q,$$

as appears by carrying out the indicated differentiations with respect to t, and if we now make use of the relations (4) and (5), our equation assumes the form

$$m\frac{d}{dt}\left(\dot{x}\frac{\partial\dot{x}}{\partial\dot{q}}+\dot{y}\frac{\partial\dot{y}}{\partial\dot{q}}+\dot{z}\frac{\partial\dot{z}}{\partial\dot{q}}\right)-m\left(\dot{x}\frac{\partial\dot{x}}{\partial q}+\dot{y}\frac{\partial\dot{y}}{\partial q}+\dot{z}\frac{\partial\dot{z}}{\partial q}\right)=Q.$$

The quantities in the two parentheses, each multiplied by m, are easily recognized as the partial derivatives with respect to $\dot{q}$ and q of the kinetic energy

$$T=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2);$$

hence the equation reduces to the form

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}}-\frac{\partial T}{\partial q}=Q, \tag{7}$$

known as the (second) *Lagrangian form* of the equation of motion of a particle constrained to a curve.

213. Particle Subject to One Condition. By Art. 200, the equations of motion are

$$m\ddot{x}=X+\lambda\phi_x,\quad m\ddot{y}=Y+\lambda\phi_y,\quad m\ddot{z}=Z+\lambda\phi_z, \tag{8}$$

with the condition

$$\phi(x, y, z, t)=0. \tag{9}$$

Let the two generalized co-ordinates q_1, q_2 be connected with the Cartesian co-ordinates by the equations

$$x=f_1(t, q_1, q_2),\quad y=f_2(t, q_1, q_2),\quad z=f_3(t, q_1, q_2). \tag{10}$$

The first of these equations gives

$$\dot{x}=\frac{\partial f_1}{\partial t}+\frac{\partial f_1}{\partial q_1}\dot{q}_1+\frac{\partial f_1}{\partial q_2}q_2;$$

hence, regarding x as a function of t, q_1, q_2, $q_1=dq_1/dt$, $\dot{q}_2=dq_2/dt$, we find:

$$\frac{\partial\dot{x}}{\partial q_1}=\frac{\partial^2 f_1}{\partial q_1\partial t}+\frac{\partial^2 f_1}{\partial {q_1}^2}\dot{q}_1+\frac{\partial^2 f_1}{\partial q_1\partial q_2}\dot{q}_2,\qquad \frac{\partial\dot{x}}{\partial\dot{q}_1}=\frac{\partial f_1}{\partial q_1}.$$

The right-hand member of the former of these equations is equivalent to $\frac{d}{dt}\frac{\partial f_1}{\partial q_1}$. As similar relations hold for y and z, we find again the relations (4) and (5), with q_1 substituted for q. It can be shown in the same way that these relations also hold if q_2 be written for q.

214. Let us now multiply the equations (8) by $\partial f_1/\partial q_1$, $\partial f_2/\partial q_1$, $\partial f_3/\partial q_1$, and add them. This gives

$$m\left(\ddot{x}\frac{\partial f_1}{\partial q_1}+\ddot{y}\frac{\partial f_2}{\partial q_1}+\ddot{z}\frac{\partial f_3}{\partial q_1}\right)=Q_1, \tag{11}$$

where $$Q_1=X\frac{\partial f_1}{\partial q_1}+Y\frac{\partial f_2}{\partial q_1}+Z\frac{\partial f_3}{\partial q_1}.$$

Similarly, multiplying (8) by $\partial f_1/\partial q_2$, $\partial f_2/\partial q_2$, $\partial f_3/\partial q_2$, and adding, we find

$$m\left(\ddot{x}\frac{\partial f_1}{\partial q_2}+\ddot{y}\frac{\partial f_2}{\partial q_2}+\ddot{z}\frac{\partial f_3}{\partial q_2}\right)=Q_2, \tag{12}$$

where $$Q_2=X\frac{\partial f_1}{\partial q_2}+Y\frac{\partial f_2}{\partial q_2}+Z\frac{\partial f_3}{\partial q_2}.$$

Each of these equations (11) and (12) can be treated by the method used in Art. 212, and we find as the final equations of motion on the surface (9) in the Lagrangian form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1}-\frac{\partial T}{\partial q_1}=Q_1, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2}-\frac{\partial T}{\partial q_2}=Q_2. \tag{13}$$

215. Free Particle. In this case three variables q_1, q_2, q_3 are required to determine the position of the particle. If the expressions of x, y, z in terms of these new variables do not contain the time explicitly, the introduction of the new variables consists merely in a change of co-ordinates. If they do contain the time, *i.e.* if we have

$$x=f_1(t, q_1, q_2, q_3), \quad y=f_2(t, q_1, q_2, q_3), \quad z=f_3(t, q_1, q_2, q_3), \tag{14}$$

the new system of co-ordinates is a moving system.

The relations (4), (5) can again be shown to hold for each of the three Lagrangian co-ordinates q_1, q_2, q_3.

If the equations of motion

$$m\ddot{x}=X, \quad m\ddot{y}=Y, \quad m\ddot{z}=Z \tag{15}$$

be multiplied by $\partial f_1/\partial q_1$, $\partial f_2/\partial q_1$, $\partial f_3/\partial q_1$ and added, we find

$$m\left(\ddot{x}\frac{\partial f_1}{\partial q_1}+\ddot{y}\frac{\partial f_2}{\partial q_1}+\ddot{z}\frac{\partial f_3}{\partial q_1}\right)=Q_1, \tag{16}$$

where $$Q_1=X\frac{\partial f_1}{\partial q_1}+Y\frac{\partial f_2}{\partial q_1}+Z\frac{\partial f_3}{\partial q_1}.$$

By the method of Art. 212, equation (16) reduces to the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1}-\frac{\partial T}{\partial q_1}=Q_1. \tag{17}$$

Similarly we find

$$\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2}-\frac{\partial T}{\partial q_2}&=Q_2,\\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_3}-\frac{\partial T}{\partial q_3}&=Q_3.\end{aligned} \tag{18}$$

The three equations (17) and (18) are the Lagrangian equations of motion of a free particle.

216. If there exists a force-function U for the forces X, Y, Z, *i.e.* if

$$X=\frac{\partial U}{\partial x}, \quad Y=\frac{\partial U}{\partial y}, \quad Z=\frac{\partial U}{\partial z},$$

we have

$$Q_1=\frac{\partial U}{\partial x}\frac{\partial f_1}{\partial q_1}+\frac{\partial U}{\partial y}\frac{\partial f_2}{\partial q_1}+\frac{\partial U}{\partial z}\frac{\partial f_3}{\partial q_1}=\frac{\partial U}{\partial q_1},$$

and similarly

$$Q_2=\frac{\partial U}{\partial q_2}, \quad Q_3=\frac{\partial U}{\partial q_3}.$$

In this case, one of the three equations (17), (18) can be replaced by the equation of kinetic energy

$$T=U+h,$$

where h is a constant.

217. The general theory of the constrained motion of a particle is treated with special care in the works of Schell, Budde, and Appell, referred to in Art. 159. In Appell's first volume, pp. 445–517, the student will find instructive examples of the application of Lagrange's equations. For more elementary problems, as also for the interesting theories of brachistochrones and tautochrones, the reader is referred, besides the works just named, to the text-books of Tait and Steele, Besant, Price, and Walton's Problems (see Art. 159).

CHAPTER VI.

KINETICS OF A RIGID BODY.

I. *General Principles.*

218. In kinetics the term *rigid body* means any system or aggregate of mass-particles whose mutual distances remain invariable. A rigid body may therefore consist of a finite number of rigidly connected particles or of a continuous mass of one, two, or three dimensions. Its motion depends not only on the forces acting on the body, but also on the way in which the mass is distributed throughout the body.

In the present section the rigid body is assumed to be free unless the contrary be stated explicitly.

219. Let us consider any one particle m of the body; at any time t, let j be its acceleration and F the resultant of all the forces acting on the particle. Then the motion of this particle (see Arts. 35, 67) is determined by the equation

$$mj = F. \tag{1}$$

It should be noticed that among the forces acting on the particle are included not only those external forces acting on the rigid body that happen to be applied at m, but also the so-called *internal forces* which would replace the rigid connection of the particle m with the rest of the body.

If, at the time t, x, y, z are the co-ordinates of the particle m with respect to a fixed set of rectangular axes, then the components of its velocity v may be denoted by $\dot{x}, \dot{y}, \dot{z}$; those of its acceleration j by $\ddot{x}, \ddot{y}, \ddot{z}$.* And if the components of F

---

* Here again we shall use this so-called *fluxional notation*, according to which derivatives with respect to the time are denoted by dots; see the foot-note to Art. 183.

along the same axes are X, Y, Z, the equation (1) can be replaced by the following three:

$$-m\ddot{x}+X=0, \quad -m\ddot{y}+Y=0, \quad -m\ddot{z}+Z=0. \qquad (2)$$

Such a set of three equations can be written down for each particle; hence, if the body consist of n particles, there would be in all $3n$ equations.

220. For the solution of particular problems these $3n$ equations are of little use, not only because their number would in general be very great and may even be infinite, but mainly because the forces X, Y, Z include the unknown reactions between the particles. It is, however, possible to deduce certain general propositions from these equations.

The $3n$ equations express the equilibrium of the system formed by *all* the forces, both internal and external, acting on the particles, and the reversed effective forces. To apply the principle of virtual work to this system, let us multiply the three equations (2) by the components δx, δy, δz of some virtual displacement of the particle m; let the same thing be done for every other particle of the body; and let all the resulting equations be added:

$$\Sigma(-m\ddot{x}+X)\delta x+\Sigma(-m\ddot{y}+Y)\delta y+\Sigma(-m\ddot{z}+Z)\delta z=0. \qquad (3)$$

221. It is important to notice that the internal reactions between the particles which make the body rigid occur in pairs of equal and opposite forces, and form, therefore, a system which is in equilibrium by itself. Hence, while these internal forces enter into the equations (2), they do not appear in equation (3), since the equal and opposite forces cancel in the summation. Thus, equation (3) expresses that *the external*

Derivatives were called *fluxes* by Newton; thus the component of the acceleration of a point in any direction is the time-flux of its velocity in that direction; the component of its effective force in any direction is the time-flux of its momentum; and so on.

forces acting on the rigid body and the reversed effective forces form a system in equilibrium; and this is **d'Alembert's Principle** for the rigid body.

It must, however, not be forgotten that the displacements δx, δy, δz should be so selected as to be compatible with the nature of the rigid body; *i.e.* with the conditions that the distances between the particles should not be disturbed.

222. The number of conditions expressing the invariability of the distances between n particles is $3n-6$. For if there were but 3 particles, the number of independent conditions would evidently be 3; for every additional particle, 3 additional conditions are required. Hence, the total number of conditions is $3+3(n-3)=3n-6$.

It follows that if a rigid body be subject to no other constraining conditions, the number of its equations of motion must be $3n-(3n-6)=6$. Hence, *a free rigid body has six independent equations of motion.* (Comp. Part I., Art. 37.)

223. The six equations of motion of the rigid body can be obtained as follows.

Imagine the equations (2), viz.

$$m\ddot{x}=X, \quad m\ddot{y}=Y, \quad m\ddot{z}=Z,$$

written down for every particle, and add the corresponding equations. This gives the first 3 of the 6 equations of motion:

$$\Sigma m\ddot{x}=\Sigma X, \quad \Sigma m\ddot{y}=\Sigma Y, \quad \Sigma m\ddot{z}=\Sigma Z. \tag{4}$$

As the internal forces cancel in the summation, the right-hand members of these equations represent the components R_x, R_y, R_z of the resultant R of all the external forces acting on the body. The left-hand members can be put into the form $d(\Sigma m\dot{x})/dt$, $d(\Sigma m\dot{y})/dt$, $d(\Sigma m\dot{z})/dt$; these are the time derivatives or fluxes of the sums of the linear momenta of all the particles parallel to the axes. The equations (4) can therefore be written in the form

$$\frac{d}{dt}\Sigma m\dot{x}=R_x, \quad \frac{d}{dt}\Sigma m\dot{y}=R_y, \quad \frac{d}{dt}\Sigma m\dot{z}=R_z. \tag{5}$$

The axes of co-ordinates are arbitrary. Hence, if we agree to call *linear momentum of the body in any direction* the algebraic sum of the linear momenta of all the particles in that direction, the equations (5) express the proposition that *the rate at which the linear momentum of a rigid body in any direction changes with the time is equal to the sum of the components of all the external forces in that direction.*

224. Let us now combine the second and third of the equations (2) by multiplying the former by z, the latter by y, and subtracting the former from the latter. If this be done for each particle, and the resulting equations be added, we find $\Sigma m(y\ddot{z}-z\ddot{y})=\Sigma(yZ-zY)$. Similarly, we can proceed with the third and first, and with the first and second of the equations (2). The result is:

$$\Sigma m(y\ddot{z}-z\ddot{y})=\Sigma(yZ-zY),\quad \Sigma m(z\ddot{x}-x\ddot{z})=\Sigma(zX-xZ),$$
$$\Sigma m(x\ddot{y}-y\ddot{x})=\Sigma(xY-yX). \tag{6}$$

Here again the internal forces disappear in the summation, so that the right-hand members are the components H_x, H_y, H_z of the vector H of the resultant couple, found by reducing all the external forces for the origin of co-ordinates. The left-hand members are the components of the resultant couple of the effective forces for the same origin.

We can also say that the right-hand members are the sums of the moments of the external forces about the co-ordinate axes (Part II., Art. 213), while the left-hand members represent the moments of the effective forces about the same axes. The latter quantities are exact derivatives, as shown in Arts. 87 and 91. The equations (6) can therefore be written in the form

$$\frac{d}{dt}\Sigma m(y\dot{z}-z\dot{y})=H_x,\quad \frac{d}{dt}\Sigma m(z\dot{x}-x\dot{z})=H_y,\quad \frac{d}{dt}\Sigma m(x\dot{y}-y\dot{x})=H_z. \tag{7}$$

As explained in Arts. 89 and 92, the quantity $m(y\dot{z}-z\dot{y})$ is called the *angular momentum* (or the moment of momentum) of the particle m about the axis of x. We may now agree to call the quantity $\Sigma m(y\dot{z}-z\dot{y})$ the *angular momentum of the body* about the axis of x, just as $\Sigma m\dot{x}$ is the linear momentum of the body along this axis; and similarly for the other axes. The meaning of the equations (7) can then be stated as follows: *The rate at which the angular momentum of a rigid body about any axis changes with the time is equal to the sum of the moments of all the external forces about this line.*

The equations (4) and (6), or (5) and (7), are the **six equations of motion of the rigid body.** The three equations (4) or (5) may be called the *equations of linear momentum,* while (6) or (7) are the *equations of angular momentum.*

225. The equations (4) and (6) can also be derived from the equation (3), which expresses d'Alembert's principle, by selecting for δx, δy, δz convenient displacements.

Thus, the rigidity of the body will evidently not be disturbed if we give to all its points equal and parallel infinitesimal displacements, since this merely amounts to subjecting the whole body to an infinitesimal translation. Equation (3) can in this case be written

$$\delta x\Sigma(-m\ddot{x}+X)+\delta y\Sigma(-m\ddot{y}+Y)+\delta z\Sigma(-m\ddot{z}+Z)=0,$$

and is therefore equivalent to the three equations (4), since δx, δy, δz are independent and arbitrary.

Again, let the body be subjected to an infinitesimal rotation of angle $\delta\theta$ about any line l.

As shown in Art. 293 of Part I., the linear velocities of any point (x, y, z) of a rigid body, due to a rotation of angular velocity $\omega=\delta\theta/\delta t$ about any line l are, if ω_x, ω_y, ω_z denote the components of ω:

$$\dot{x}=\omega_y z-\omega_z y,\quad \dot{y}=\omega_z x-\omega_x z,\quad \dot{z}=\omega_x y-\omega_y x.$$

Hence, putting $\omega_x\delta t=\delta\theta_x$, $\omega_y\delta t=\delta\theta_y$, $\omega_z\delta t=\delta\theta_z$, we have for the displacements of the point (x, y, z), due to a rotation of angle $\delta\theta$,

$$\delta x=z\,\delta\theta_y-y\,\delta\theta_z,\quad \delta y=x\,\delta\theta_z-z\,\delta\theta_x,\quad \delta z=y\,\delta\theta_x-x\,\delta\theta_y.$$

If these values be introduced into d'Alembert's equation (3) and the terms in $\delta\theta_x$, $\delta\theta_y$, $\delta\theta_z$ be collected, it assumes the form

$$\delta\theta_x\Sigma[-m(y\ddot{z}-z\ddot{y})+yZ-zY]+\delta\theta_y\Sigma[-m(z\ddot{x}-x\ddot{z})+zX-xZ]$$
$$+\delta\theta_z\Sigma[-m(x\ddot{y}-y\ddot{x})+xY-yX]=0;$$

as $\delta\theta_x$, $\delta\theta_y$, $\delta\theta_z$ are independent and arbitrary, their coefficients must vanish separately, and this gives the equations (6).

226. The equations of linear momentum, (4) or (5), admit of a further simplification, owing to the fundamental property of the centroid. By Part II., Art. 13, the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centroid satisfy the relations

$$M\bar{x}=\Sigma mx,\quad M\bar{y}=\Sigma my,\quad M\bar{z}=\Sigma mz,$$

where $M=\Sigma m$ is the whole mass of the body. Differentiating these equations, we find

$$M\dot{\bar{x}}=\Sigma m\dot{x},\quad M\dot{\bar{y}}=\Sigma m\dot{y},\quad M\dot{\bar{z}}=\Sigma m\dot{z},$$

and

$$M\ddot{\bar{x}}=\Sigma m\ddot{x},\quad M\ddot{\bar{y}}=\Sigma m\ddot{y},\quad M\ddot{\bar{z}}=\Sigma mz,$$

where $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ are the components of the velocity $\bar{v}$, and $\ddot{\bar{x}}$, $\ddot{\bar{y}}$, $\ddot{\bar{z}}$ those of the acceleration $\bar{j}$, of the centroid.

The equations (4) or (5) can therefore be reduced to the form

$$M\ddot{\bar{x}}=\frac{d}{dt}M\dot{\bar{x}}=R_x,\quad M\ddot{\bar{y}}=\frac{d}{dt}M\dot{\bar{y}}=R_y,\quad M\ddot{\bar{z}}=\frac{d}{dt}M\dot{\bar{z}}=R_z,\qquad (8)$$

whence

$$M\bar{j}=\frac{d}{dt}M\bar{v}=R\,;\qquad (9)$$

i.e. if the whole mass of the body be regarded as concentrated at the centroid, the effective force of the centroid, or the time-rate of change of its momentum, is equal to the resultant of all the external forces. It follows that *the centroid of a rigid body moves as if it contained the whole mass, and all the external forces were applied at this point parallel to their original directions.*

227. If, in particular, the resultant R vanish (while there may be a couple H acting on the body), we have by (8) and (9) $\bar{j}=0$; hence $\bar{v}=$ const.; i.e. *if the resultant force be zero, the centroid moves uniformly in a straight line.*

This proposition, which can also be expressed by saying that, if $R=0$, the momentum $M\bar{v}$ of the centroid remains constant, or, using the form (5) of the equations of motion, that the linear momentum of the body in any direction is constant, is known as the **principle of the conservation of linear momentum,** or the principle of the conservation of the motion of the centroid.

228. Let us next consider the equations of angular momentum, (6) or (7). To introduce the properties of the centroid, let us put $x-\bar{x}=\xi$, $y-\bar{y}=\eta$, $z-\bar{z}=\zeta$, so that ξ, η, ζ are the co-ordinates of the point (x, y, z) with respect to parallel axes through the centroid. The substitution of $x=\bar{x}+\xi$, $y=\bar{y}+\eta$, $z=\bar{z}+\zeta$ and their derivatives into the expression $y\dot{z}-z\dot{y}$ gives

$$\bar{y}\dot{\bar{z}}-\bar{z}\dot{\bar{y}}+\bar{y}\dot{\zeta}-\bar{z}\dot{\eta}+\eta\dot{\bar{z}}-\zeta\dot{\bar{y}}+\eta\dot{\zeta}-\zeta\dot{\eta}.$$

To form $\Sigma m(y\dot{z}-z\dot{y})$ we must multiply by m and sum throughout the body; in this summation, $\bar{y}$, $\bar{z}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ are constant and, by the property of the centroid, $\Sigma m\eta=0$, $\Sigma m\zeta=0$, $\Sigma m\dot{\eta}=0$, $\Sigma m\dot{\zeta}=0$. Hence we find

$$\Sigma m(y\dot{z}-z\dot{y})=\Sigma m(\eta\dot{\zeta}-\zeta\dot{\eta})+M(\bar{y}\dot{\bar{z}}-\bar{z}\dot{\bar{y}}).$$

The second term in the right-hand member is the angular momentum of the centroid about the axis of x (the whole mass M of the body being regarded as concentrated at this point), while the first term is the angular momentum of the body about a parallel to the axis of x, drawn through the centroid.

Similar relations hold for the angular momenta about the axes of y and z; and as these axes are arbitrary, we conclude that *the angular momentum of a rigid body about any line is equal to its angular momentum about a parallel through the centroid plus the angular momentum of the centroid about the former line.*

229. Differentiating the above expression, we find

$$\frac{d}{dt}\Sigma m(y\dot{z}-z\dot{y})=\frac{d}{dt}\Sigma m(\eta\dot{\zeta}-\zeta\dot{\eta})+M(\bar{y}\ddot{\bar{z}}-\bar{z}\ddot{\bar{y}}).$$

The first of the equations (7) can therefore be written

$$\frac{d}{dt}\Sigma m(\eta\dot{\zeta}-\zeta\dot{\eta})+M(\bar{y}\ddot{\bar{z}}-\bar{z}\ddot{\bar{y}})=H_x.$$

Now, if at any time t the centroid were taken as origin, so that $\bar{y}=0$, $\bar{z}=0$, this equation would reduce to the form

$$\frac{d}{dt}\Sigma m(\eta\dot{\zeta}-\zeta\dot{\eta})=H_x,$$

which is entirely independent of the co-ordinates of the centroid. On the other hand, wherever the origin is taken, if the centroid were a fixed point, the same equation would be obtained.

Similar considerations apply of course to the other two equations (7). It follows that *the motion of a rigid body relative to the centroid is the same as if the centroid were fixed.*

230. If, in particular, the resultant couple H be zero for any particular origin O (which will be the case not only when all external forces are zero, but also when the directions of all forces pass through the point O), the equations (7) can be integrated and give

$$\Sigma m(y\dot{z}-z\dot{y})=C_1,\quad \Sigma m(z\dot{x}-x\dot{z})=C_2,\quad \Sigma m(x\dot{y}-y\dot{x})=C_3, \quad (10)$$

where C_1, C_2, C_3 are constants of integration (comp. Art. 94). Hence, *if the external forces pass through a fixed point, the angular momentum of the body about any line through this point is constant; if there are no external forces, the angular momentum is constant for any line whatever.* This is the **principle of the conservation of angular momentum.**

231. Another interpretation can be given to these equations. As shown in Arts. 88 and 91, the quantities $y\dot{z}-z\dot{y}$, $z\dot{x}-x\dot{z}$, $x\dot{y}-y\dot{x}$ can be regarded as sectorial velocities. Thus, if the radius vector, drawn from the origin to the particle m, be projected on the yz-plane, $y\dot{z}-z\dot{y}$ is twice the sectorial velocity of this radius vector in the yz-plane, $\frac{1}{2}(ydz-zdy)$ being the elementary sector described in the element of time dt. Let us denote by dS_x the sum of all these elementary sectors for the various particles, each multiplied by the mass of the particle; and similarly by dS_y, dS_z the corresponding sums of the projections on the other co-ordinate planes. Then the equations (10) can be written in the form

$$2\,\dot{S}_x = C_1,\quad 2\,\dot{S}_y = C_2,\quad 2\,\dot{S}_z = C_3. \tag{11}$$

Hence the proposition of Art. 230 might be called the principle of the conservation of sectorial velocities; it is more commonly called the **principle of the conservation of areas.**

The equations (11) can be integrated again and give, if the sectors be measured from the positions of the radii vectores at the time $t=0$,

$$S_x = \tfrac{1}{2}\,C_1 t,\quad S_y = \tfrac{1}{2}\,C_2 t,\quad S_z = \tfrac{1}{2}\,C_3 t.$$

232. If the radii vectores be projected on any plane through the origin whose normal has the direction cosines α, β, γ, the sum of the elementary sectors described in this plane, each multiplied by the mass, will be

$$dS = \tfrac{1}{2}(C_1\alpha + C_2\beta + C_3\gamma)dt\,;$$

hence
$$S = \tfrac{1}{2}(C_1\alpha + C_2\beta + C_3\gamma)t.$$

On the other hand, by (10), the angular momentum of the body about the normal of this plane has the expression $C_1\alpha + C_2\beta + C_3\gamma$, as it must be equal to the sum of the projections on this normal of the angular momenta about the axes of co-ordinates, which can be regarded as vectors laid off on these axes.

Now it is easy to see that this angular momentum $C_1\alpha + C_2\beta + C_3\gamma$, and hence the quantity S for a given time t, is greatest for the diagonal of the parallelopiped, whose edges are equal to C_1, C_2, C_3 along the axes, *i.e.* for the normal to the plane

$$C_1 x + C_2 y + C_3 z = 0. \tag{12}$$

For, the direction cosines of this normal are $\alpha' = C_1/D$, $\beta' = C_2/D$, $\gamma' = C_3/D$, where $D = \sqrt{C_1^2 + C_2^2 + C_3^2}$; and the quantity $C_1\alpha + C_2\beta + C_3\gamma$ can be put into the form

$$D\left(\frac{C_1}{D}\alpha + \frac{C_2}{D}\beta + \frac{C_3}{D}\gamma\right) = D(\alpha'\alpha + \beta'\beta + \gamma'\gamma),$$

where the quantity in parenthesis is the cosine of the angle between the directions $(\alpha', \beta', \gamma')$ and (α, β, γ), and is therefore greatest when these directions coincide.

The plane (12) about whose normal the angular momentum is greatest, and by projection on which the area S is made greatest, is called Laplace's **invariable plane.** As its equation is independent of t, it remains fixed. The normal of this plane is sometimes called the *invariable line or direction.*

233. Let us now return to the general case of the motion of a rigid body acted upon by any forces whatever.

The propositions of Arts. 226 and 229 together establish the so-called **principle of the independence of the motions of translation and rotation.** In studying the motion of a rigid body it is possible, according to this principle, to consider separately the motion of translation of the centroid, and the rotation of the body about the centroid.

By Art. 226, the motion of the centroid is the same as that of a particle of mass M acted upon by all the external forces transferred parallel to themselves to the centroid. As the motion of a particle has been discussed in Chapter V., nothing further need be said about this part of the problem.

By Art. 229, the motion of the body about the centroid is the same as if the centroid were fixed. The problem of the motion of a rigid body with a fixed point is therefore of great importance; it will be discussed in Section IV. The more simple special case of a rigid body with a fixed axis is treated in Section III. The solution of both these problems depends on the equations (6) or (7).

234. In d'Alembert's equation (3) it is of course allowable to substitute for the virtual displacements δx, δy, δz the actual displacements dx, dy, dz of the particles in any motion of a free rigid body, since these actual displacements are certainly compatible with the condition of rigidity. The equation can then be written

$$\Sigma m(\ddot{x}dx+\ddot{y}dy+\ddot{z}dz)=\Sigma(Xdx+Ydy+Zdz). \qquad (13)$$

The left-hand member of this equation evidently represents the differential of the kinetic energy

$$T=\Sigma\tfrac{1}{2}mv^2=\Sigma\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \tag{14}$$

of the body, while the right-hand member is the elementary work of the external forces. Hence equation (13) expresses the **principle of kinetic energy** for a free rigid body, viz. the proposition that, *for any infinitesimal displacement of the body, the increase of the kinetic energy is equal to the sum of the works done by all the external forces.*

235. By introducing the co-ordinates of the centroid, *i.e.* by putting $x=\bar{x}+\xi$, $y=\bar{y}+\eta$, $z=\bar{z}+\zeta$, as in Art. 228, the expression for the kinetic energy assumes the form (since $\Sigma m\dot{\xi}=0$, $\Sigma m\dot{\eta}=0$, $\Sigma m\dot{\zeta}=0$):

$$\begin{aligned} T&=\Sigma\,\tfrac{1}{2}\,m(\dot{\bar{x}}^2+\dot{\bar{y}}^2+\dot{\bar{z}}^2)+\Sigma\,\tfrac{1}{2}\,m(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)\\ &=\tfrac{1}{2}\,M\bar{v}^2+\Sigma\,\tfrac{1}{2}\,mu^2, \end{aligned} \tag{15}$$

where $\bar{v}$ is the velocity of the centroid and u the *relative* velocity of any particle m with respect to the centroid.

Thus, it appears that *the kinetic energy of a free rigid body consists of two parts, one of which is the kinetic energy of the centroid* (the whole mass being regarded as concentrated at this point), *while the other may be called the relative kinetic energy with respect to the centroid.*

236. By the same substitution the right-hand member of equation (13), *i.e.* the elementary work $\Sigma(Xdx+Ydy+Zdz)$, resolves itself into the two parts

$$(d\bar{x}\,\Sigma X+d\bar{y}\,\Sigma Y+d\bar{z}\,\Sigma Z)+\Sigma(Xd\xi+Yd\eta+Zd\zeta).$$

The first parenthesis contains the work that would be done by all the external forces if they were applied at the centroid; it is therefore equal to the kinetic energy of the centroid, that is to $d(\frac{1}{2}\,M\bar{v}^2)$. The equation of kinetic energy (13) reduces, therefore, to the following:

$$d(\Sigma\,\tfrac{1}{2}\,mu^2)=\Sigma(Xd\xi+Yd\eta+Zd\zeta)\,; \tag{16}$$

in other words, *the principle of kinetic energy holds for the relative motion with respect to the centroid.*

237. Impulses. The equations determining the effect of a system of impulses (see Arts. 2–5) on a rigid body are readily obtained from the general equations of motion (4) and (6). We shall denote the impulse of a force F by $\underset{\cdot}{F}$. It will be remembered that the impulse $\underset{\cdot}{F}$ of a force F is its time integral; *i.e.*

$$\underset{\cdot}{F}=\int_t^{t'} F dt.$$

We confine ourselves to the case when $t'-t$ is very small and F very large, in which case the action of the impulsive force F is measured by its impulse $\underset{\cdot}{F}$.

If all the forces acting on a rigid body are of this nature, and the impulses of X, Y, Z during the short interval $t'-t$ be denoted by $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$, the integration of the equations (4) from $t=t$ to $t=t'$ gives

$$\Sigma m(\dot{x}'-\dot{x})=\Sigma \underset{\cdot}{X},\quad \Sigma m(\dot{y}'-\dot{y})=\Sigma \underset{\cdot}{Y},\quad \Sigma m(\dot{z}'-\dot{z})=\Sigma \underset{\cdot}{Z}, \qquad (17)$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$ denote the velocities of the particle m at the time t just before the impulse, and $\dot{x}'$, $\dot{y}'$, $\dot{z}'$ those at the time t' just after the action of the impulse.

Similarly the equations (6) give

$$\begin{aligned}
\Sigma m[y(\dot{z}'-\dot{z})-z(\dot{y}'-\dot{y})]&=\Sigma(y\underset{\cdot}{Z}-z\underset{\cdot}{Y}),\\
\Sigma m[z(\dot{x}'-\dot{x})-x(\dot{z}'-\dot{z})]&=\Sigma(z\underset{\cdot}{X}-x\underset{\cdot}{Z}),\qquad (18)\\
\Sigma m[x(\dot{y}'-\dot{y})-y(\dot{x}'-\dot{x})]&=\Sigma(x\underset{\cdot}{Y}-y\underset{\cdot}{X}).
\end{aligned}$$

238. In determining the effect on a rigid body of a system of such impulses, any ordinary forces acting on the body at the same time are neglected because the changes of velocity produced by them during the very short time τ are small in comparison with the changes of velocity $\dot{x}'-\dot{x}$, $\dot{y}'-\dot{y}$, $\dot{z}'-\dot{z}$ produced by the impulses. For the mathematical treatment it is generally most convenient to define the impulse $\underset{\cdot}{F}$ of an impulsive force F as the limit of the integral $\int_t^{t'} F dt$ when $t'-t$ approaches 0 and

F approaches ∞ (Art. 5); in this case it is strictly true that the effect of ordinary forces can be neglected when impulsive forces act on the body.

If the rigid body be originally at rest, it will be convenient to denote by $\dot{x}$, $\dot{y}$, $\dot{z}$ the components of the velocity of the particle m just after the action of the impulses. We may also denote by $\underset{.}{R}$ the resultant of all the impulses, by $\underset{.}{H}$ the resultant impulsive couple for the reduction to the origin of coordinates, and mark the components of $\underset{.}{R}$ and $\underset{.}{H}$ by subscripts, as in the case of forces. With these notations the effect of a system of impulses on a body at rest is given by the equations

$$\Sigma m\dot{x} = \underset{.}{R}_x, \quad \Sigma m\dot{y} = \underset{.}{R}_y, \quad \Sigma m\dot{z} = \underset{.}{R}_z, \tag{19}$$

$$\Sigma m(y\dot{z} - z\dot{y}) = \underset{.}{H}_x, \quad \Sigma m(z\dot{x} - x\dot{z}) = \underset{.}{H}_y, \quad \Sigma m(x\dot{y} - y\dot{x}) = \underset{.}{H}_z. \tag{20}$$

In the equations (19) we have, of course, $\Sigma m\dot{x} = M\dot{\bar{x}}$, $\Sigma m\dot{y} = M\dot{\bar{y}}$, $\Sigma m\dot{z} = M\dot{\bar{z}}$, where $\dot{\bar{x}}, \dot{\bar{y}}, \dot{\bar{z}}$ are the components of the velocity of the centroid, and M is the mass of the body; i.e. *the momentum of the centroid is equal to the resultant impulse*. The meaning of the equations (20) can be stated by saying that *the angular momentum of the body about any axis is equal to the moment of all the impulses about the same axis.*

II. *Moments of Inertia and Principal Axes.*

I. INTRODUCTION.

239. As will be shown in Sections III. and IV., the rotation of a rigid body about any axis depends not only on the forces acting on the body, but also on the way in which the mass is distributed throughout the body. This distribution of mass is characterized by the position of the centroid and by that of certain lines in the body called *principal axes.*

It has been shown in Part II., Art. 13, that the centroid of a mass is found by determining the *moments,* or more precisely, the *moments of the first order,* Σmx, Σmy, Σmz, of the mass with respect to the co-ordinate planes, *i.e.* the sums of all mass-particles m each multiplied by its distance from the co-ordinate plane.

The principal axes of a mass or body can be found by determining the *moments of the second order,* Σmx^2, Σmy^2, Σmz^2, Σmyz, Σmzx, Σmxy of the mass with respect to the same planes. We proceed, therefore, to study the theory of such moments.

240. If in a rigid body the mass m of each particle be multiplied by the square of its distance r from a given point, plane, or line, the sum

$$\Sigma mr^2 = m_1r_1^2 + m_2r_2^2 + \cdots,$$

extended over the whole body, is called the *quadratic moment,* or, more commonly, the **moment of inertia** of the body for that point, plane, or line.

If the body is not composed of discrete particles, but forms a continuous mass of one, two, or three dimensions, this mass can be resolved into elements of mass dm, and the sum Σmr^2 becomes a single, double, or triple integral $\int r^2dm$.

Expressions of the form $\Sigma m r_1 r_2$, or $\int r_1 r_2 dm$, where r_1, r_2 are the distances of m or of dm from two planes (usually at right angles), are called *moments of deviation* or **products of inertia.**

241. The determination of the moment of inertia of a continuous mass is a mere problem of integration; the methods are similar to those for finding the moments of mass of the first order required for determining centroids (see Part II., Chapter III.), the only difference being that each element of mass must be multiplied by the square, instead of the first power, of the distance.

A moment of inertia is not a directed quantity; it is not a vector, but a scalar; indeed, it is a positive quantity, provided the masses are all positive, as we shall here assume.

The moment of inertia of any number of bodies or masses for any given point, plane, or line is obviously the sum of the moments of inertia of the separate bodies or masses for the same point, plane, or line.

242. The moment of inertia $\Sigma m r^2$ of any body whose mass is $M = \Sigma m$ can always be expressed in the form

$$\Sigma m r^2 = M \cdot r_0^2,$$

where r_0 is a length called the **radius of inertia**, arm of inertia, or *radius of gyration.* This length r_0 is evidently a kind of average value of the distances r, its value being intermediate between the greatest r' and least r'' of these distances r. For we have $\Sigma m r'^2 > \Sigma m r^2 > \Sigma m r''^2$, or, since $\Sigma m r'^2 = M r'^2$, $\Sigma m r^2 = M r_0^2$, $\Sigma m r''^2 = M r''^2$,

$$r' > r_0 > r''.$$

243. As an example, let us determine the moment of inertia of a *homogeneous rectilinear segment* (straight rod or wire of constant cross-section and density) for its middle point (or, what amounts to the same thing, for a line or plane through this point at right angles to the segment).

Let $2l$ be the length of the rod (Fig. 28), O its middle point, ρ its density (*i.e.* the mass of unit length), x the distance OP

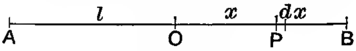

Fig. 28.

of any element $dm=\rho dx$ from the middle point. Then we have, for the moment of inertia I,

$$I=\int_{-l}^{l} x^2\cdot\rho dx=\tfrac{2}{3}\rho l^3,$$

and for the radius of inertia r_0, since the whole mass is $M=2\rho l$,

$$r_0^2=\frac{I}{M}=\tfrac{1}{3}l^2.$$

244. Exercises.

Determine the radius of inertia in the following cases. When nothing is said to the contrary, the masses are supposed to be homogeneous.

(1) Segment of straight line of length l, for a perpendicular through one end.

(2) Rectangular area of length l and width h: (*a*) for the side h; (*b*) for the side l; (*c*) for a line through the centroid parallel to the side h; (*d*) for a line through the centroid parallel to the side l.

(3) Triangular area of base b and height h, for a line through the vertex parallel to the base.

(4) Square of side a, for a diagonal.

(5) Regular hexagon, for a diagonal.

(6) Right cylinder or prism of height $2h$, for the plane bisecting the height at right angles.

(7) Segment of straight line of length l, for one end, when the density is proportional to the nth power of the distance from this end. Deduce from this: (*a*) the result of Ex. (1); (*b*) that of Ex. (3); (*c*) the radius of inertia of a homogeneous pyramid or cone (right or oblique) of height h, for a plane through the vertex parallel to the base.

(8) Circular area (plate, disc, lamina) of radius a, for any diameter.

(9) Circular line (wire) of radius a, for a diameter.

(10) Solid sphere, for a diametral plane.

(11) Solid ellipsoid, for the three principal planes.

(12) Area of ring bounded by concentric circles of radii a_1, a_2, for a diameter.

(13) Area of the cross-section of a ⊥-iron: (*a*) for its line of symmetry; (*b*) for its base. (Dimensions as in Fig. 8, Part II., p. 18.)

(14) A rectangular door of width b and height h has a thickness δ to a distance a from the edges, while the rectangular panel (whose dimensions are $b - 2a$, $h - 2a$) has half this thickness. Find the moment of inertia for a line through the centroid parallel to the side b.

245. The moment of inertia of any mass M *for a point* can easily be found if the moments of inertia of the same mass are known for any line passing through the point, and for the plane through the point perpendicular to the line. Let O (Fig. 29) be the point, l the line, π the plane; r, q, p the perpendicular distances of any particle of mass m from O, l, π, respectively. Then we have, evidently, $r^2 = q^2 + p^2$. Hence, multiplying by m, and summing over the whole mass M,

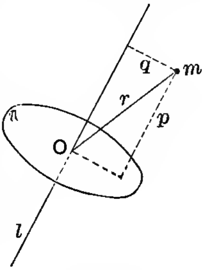

Fig. 29.

$$\Sigma m r^2 = \Sigma m q^2 + \Sigma m p^2; \qquad (1)$$

or, putting $\Sigma m r^2 = M r_0^2$, $\Sigma m q^2 = M q_0^2$, $\Sigma m p^2 = M p_0^2$, where r_0, q_0, p_0 are the radii of inertia for O, l, π,

$$r_0^2 = q_0^2 + p_0^2. \qquad (1')$$

246. The moment of inertia of any mass M *for a line* is equal to the sum of the moments of inertia of the same mass for any two rectangular planes passing through the line. Thus, in particular, the moment of inertia for the axis of x in a rectangular system of co-ordinates is equal to the sum of the moments of inertia for the zx-plane and xy-plane. This follows at once by considering that the square of the distance of any point from the line is equal to the sum of the squares of the distances of the same point from the two planes. Thus, if q be the distance of any point (x, y, z) from the axis of x, we have $q^2=y^2+z^2$; whence

$$\Sigma mq^2=\Sigma my^2+\Sigma mz^2.$$

247. It follows, from the last article, that *the moment of inertia* I_x *of a plane area, for any line perpendicular to its plane,* is

$$I_x=I_y+I_z,$$

if I_y, I_z are the moments of inertia of the area for any two rectangular lines in the plane through the foot of the perpendicular line.

248. The problem of *finding the moment of inertia of a given mass for a line* l′, *when it is known for a parallel line* l, is of great importance.

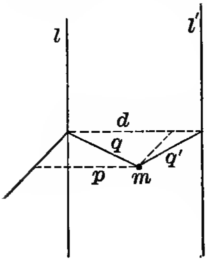

Fig. 30.

Let Σmq^2 be the moment of inertia of the given mass for the line l (Fig. 30), $\Sigma mq'^2$ that for a parallel line l' at the distance d from l. The distances q, q' of any particle m from l, l' form with d a triangle which gives the relation

$$q'^2=q^2+d^2-2qd\cos(q, d).$$

Multiplying by m, and summing over the whole mass M, we find

$$\Sigma mq'^2=\Sigma mq^2+Md^2-2d\Sigma mq\cos(q, d).$$

Now the figure shows that the product $q\cos(q, d)$ in the last term is the distance p of the particle m from a plane through l at right angles to the plane determined by l and l'. We have, therefore,

$$\Sigma mq'^2=\Sigma mq^2+Md^2-2d\Sigma mp, \tag{2}$$

where the last term contains the moment of the first order $\Sigma mp=M\bar{p}$ of the given mass M for the plane just mentioned.

If, in particular, this plane contains the centroid G of the mass M, we have $\Sigma mp=0$, so that the formula reduces to

$$\Sigma mq'^2=\Sigma mq^2+Md^2. \tag{3}$$

Introducing the radii of inertia q_0', q_0, this can be written

$$q_0'^2=q_0^2+d^2. \tag{3'}$$

249. Similar considerations hold for the moments of inertia Σmp^2, $\Sigma mp'^2$ with respect to two parallel planes π, π' at the distance d from each other. We have, in this case, $p'=p-d$; hence,

$$\Sigma mp'^2=\Sigma mp^2+Md^2-2d\Sigma mp, \tag{4}$$

and if the plane π contain the centroid G,

$$\Sigma mp'^2=\Sigma mp^2+Md^2. \tag{5}$$

250. Of special importance is the case in which one of the lines (or planes), say l (π), contains the centroid. The formulæ (3), (3′), and (5) hold in this case; and if we agree to designate any line (plane) passing through the centroid as a *centroidal* line (plane), our proposition can be expressed as follows: *The moment of inertia for any line (plane) is found from the moment of inertia for the parallel centroidal line (plane) by adding to the latter the product* Md² *of the whole mass into the square of the distance of the lines (planes).*

It will be noticed that of all parallel lines (planes) the centroidal line (plane) has the least moment of inertia.

251. Exercises.

Determine the radius of inertia of the following homogeneous masses:

(1) Rectangular plate of length l and width h, for a centroidal line perpendicular to its plane.

(2) Area of equilateral triangle of side a: (a) for a centroidal line parallel to the base; (b) for an altitude; (c) for a centroidal line perpendicular to its plane.

(3) Circular disc of radius a: (a) for a tangent; (b) for a line through the centre perpendicular to the plane of the disc; (c) for a perpendicular to its plane through a point in the circumference.

(4) Solid sphere, for a diameter.

(5) Area of ring bounded by concentric circles of radii a_1, a_2, for a line through the centre perpendicular to the plane of the ring. For a ring whose thickness $a_2 - a_1$ is infinitesimal, the result can also be obtained by differentiation from Ex. (3) (*b*).

(6) Spherical shell of infinitesimal thickness, for a diameter.

(7) Right circular cylinder, of radius a and height $2h$: (*a*) for its axis; (*b*) for a generating line; (*c*) for a centroidal line in the middle cross-section.

(8) Prove that, in a right prism or cylinder of any cross-section, we have $q^2 = q_a^2 + q_c^2$, where q is the radius of inertia of the prism or cylinder for a line bisecting the axis at right angles, q_a the radius of inertia of the axis, q_c that of the middle cross-section, for the same line.

(9) Area of ellipse: (*a*) for the major axis; (*b*) for the minor axis; (*c*) for the perpendicular to its plane through the centre.

(10) Solid ellipsoid, for each of the three axes.

(11) Area of the cross-section of a ⊥-iron, for a centroidal line parallel to the flange. (Compare Art. 244, Ex. (13).)

(12) Area of the cross-section of a symmetrical double T-iron, width of flanges b, thickness of flanges δ, height of web h, thickness of web 2δ; for the two axes of symmetry, and for a centroidal line perpendicular to its plane.

(13) Wire bent into an equilateral triangle of side a, for a centroidal line at right angles to the plane of the triangle.

252. Routh's Rule. In the case of homogeneous masses with axes of symmetry, the radius of inertia for an axis of symmetry can readily be derived by the following mnemonical rule: *The square of the radius of inertia is* $\frac{1}{3}$, $\frac{1}{4}$, *or* $\frac{1}{5}$ *of the sum of the squares of the perpendicular semi-axes, according as the mass is rectangular, elliptic, or ellipsoidal.*

The proof rests on the following typical cases which are easily proved directly (comp. Art. 251, Ex. (1), (9), (10)):

(1) Rectangular area whose sides are $2a$, $2b$, for a centroidal line perpendicular to its plane: $q^2 = \frac{1}{3}(a^2 + b^2)$.

(2) Elliptic area whose axes are $2a$, $2b$, for a centroidal line perpendicular to its plane: $q^2 = \frac{1}{4}(a^2 + b^2)$.

(3) Solid ellipsoid whose axes are $2a$, $2b$, $2c$, for its axes:

$$q_a^2 = \tfrac{1}{5}(b^2 + c^2), \quad q_b^2 = \tfrac{1}{5}(c^2 + a^2), \quad q_c^2 = \tfrac{1}{5}(a^2 + b^2).$$

A large number of special cases can be brought under this rule, as will be seen from the following exercises. It should be remembered that the radius of inertia of a homogeneous right prism or cylinder for its axis is the same as that of its cross-section.

253. Exercises. Apply Routh's rule to find the radius of inertia in the following cases:

(1) Solid sphere of radius a, for a diameter.

(2) Right circular cylinder, for its axis.

(3) Thin straight rod of length $2a$, for a perpendicular through its middle point.

(4) Rectangular disc whose sides are $2a$, $2b$, for a line in its plane bisecting the sides $2a$.

(5) Circular disc, for a diameter.

2. ELLIPSOIDS OF INERTIA.

254. The moments of inertia of a given mass for the different lines of space are not independent of each other. Several examples of this have already been given. It has been shown, in particular (Art. 248), that if the moment of inertia be known for any line, it can be found for any parallel line. It follows that if the moments be known for all lines through any given point, the moments for all lines of space can be found. We now proceed to study the relations between the moments of inertia for all the lines passing through any given point O.

255. It will here be convenient to refer the given mass M to a rectangular system of co-ordinates with the origin at the point O. Let x, y, z be the co-ordinates of any particle m of the mass; and let us denote by A, B, C the moments of inertia of M for the axes of x, y, z; by A', B', C' those for the planes yz, zx, xy; by D, E, F the products of inertia (Art. 240) for the co-ordinate planes; *i.e.* let us put:

$$\begin{aligned} A &= \Sigma m\,(y^2+z^2), & A' &= \Sigma m x^2, & D &= \Sigma m yz, \\ B &= \Sigma m\,(z^2+x^2), & B' &= \Sigma m y^2, & E &= \Sigma m zx, \\ C &= \Sigma m\,(x^2+y^2), & C' &= \Sigma m z^2, & F &= \Sigma m xy. \end{aligned} \qquad (6)$$

256. These nine quantities are not independent of each other. We have evidently

$$A = B' + C', \qquad B = C' + A', \qquad C = A' + B';$$

hence, solving for A', B', C',

$$A' = \tfrac{1}{2}(B+C-A), \qquad B' = \tfrac{1}{2}(C+A-B), \qquad C' = \tfrac{1}{2}(A+B-C).$$

The moment of inertia for the origin O is

$$\Sigma m r^2 = \Sigma m\,(x^2+y^2+z^2) = A' + B' + C' = \tfrac{1}{2}(A+B+C). \qquad (7)$$

257. The moment of inertia I for any line through O can be expressed by means of the six quantities A, B, C, D, E, F; and the moment of inertia I' for any plane through O can be expressed by means of A', B', C', D, E, F.

Let π (Fig. 31) be any plane passing through O; l its normal; α, β, γ the direction cosines of l; and, as before (Art. 245), p, q, r the distances of any point (x, y, z) of the given mass from π, l, and O, respectively. Then, projecting the closed polygon formed by r, x, y, z on the line l, we have

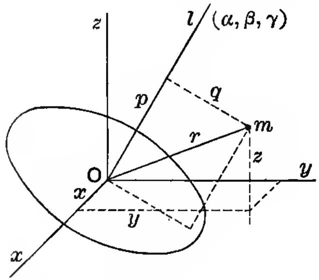

Fig. 31.

$$p = \alpha x + \beta y + \gamma z;$$

hence, squaring, multiplying by m, and summing over the whole mass, we find

$$\Sigma m p^2 = \alpha^2 \Sigma m x^2 + \beta^2 \Sigma m y^2 + \gamma^2 \Sigma m z^2 + 2\,\beta\gamma \Sigma m yz + 2\,\gamma\alpha \Sigma m zx + 2\,\alpha\beta \Sigma m xy,$$

or, with the notations (6),

$$I' = A'\alpha^2 + B'\beta^2 + C'\gamma^2 + 2\,D\beta\gamma + 2\,E\gamma\alpha + 2\,F\alpha\beta. \qquad (8)$$

Thus *the moment of inertia for any plane through the origin is expressed as a homogeneous quadratic function of the direction cosines of the normal of the plane.*

258. The moment of inertia $I=\Sigma mq^2$ for the line l can now be found from equation (1), Art. 245, by substituting for Σmr^2 and Σmp^2 their values from (7) and (8):

$$I=\Sigma mr^2-I'=A'+B'+C'-I'$$
$$=A'(1-\alpha^2)+B'(1-\beta^2)+C'(1-\gamma^2)-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta,$$

or, since $\alpha^2+\beta^2+\gamma^2=1$,

$$I=A'(\beta^2+\gamma^2)+B'(\gamma^2+\alpha^2)+C'(\alpha^2+\beta^2)-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta$$
$$=\alpha^2(B'+C')+\beta^2(C'+A')+\gamma^2(A'+B')-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta;$$

hence, finally, applying the relations of Art. 256,

$$I=A\alpha^2+B\beta^2+C\gamma^2-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta. \tag{9}$$

The moment of inertia for any line through the origin is, therefore, also *a homogeneous quadratic function of the direction cosines of the line.*

259. These results suggest a geometrical interpretation. Imagine an arbitrary length $OP=\rho$ laid off from the origin O on the line l whose direction cosines are α, β, γ; the co-ordinates of the extremity P of this length will be $x=\rho\alpha$, $y=\rho\beta$, $z=\rho\gamma$. Now, if equation (9) be multiplied by ρ^2, it assumes the form

$$Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=\rho^2 I,$$

which represents a quadratic surface provided that ρ be so selected for the different lines through O as to make $\rho^2 I$ constant, say $\rho^2 I=\kappa^2$. Hence, *if on every line* l *through the origin a length* $OP=\rho=\kappa/\sqrt{I}$ *be laid off,* i.e. *a length inversely proportional to the square root of the moment of inertia* I *for this line* l, *the points* P *will lie on the quadric surface*

$$Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=\kappa^2.$$

The constant κ^2 may be selected arbitrarily; to preserve the homogeneity of the equation it will be convenient to put it into the form $\kappa^2 = M\epsilon^4$, where ϵ is still arbitrary.

260. As moments of inertia are essentially positive quantities, the radii vectores of the surface

$$Ax^2 + By^2 + Cz^2 - 2\,Dyz - 2\,Ezx - 2\,Fxy = M\epsilon^4 \tag{10}$$

are all real, and the surface is an ellipsoid. It is called the *ellipsoid of inertia*, or the **momental ellipsoid**, of the point O. This point O is the centre; the axes of the ellipsoid are called the **principal axes** at the point O; and the moments of inertia for these axes are called the *principal moments of inertia* at the point O. Among these there will evidently be the greatest and least of all the moments of the point O, the greatest moment corresponding to the shortest, the least to the longest axis of the ellipsoid.

It may be observed that, owing to the relations of Art. 256, which show that the sum of any two of the quantities A, B, C is always greater than the third, not every ellipsoid can be regarded as the momental ellipsoid of some mass. An ellipsoid can be a momental ellipsoid only when a triangle can be constructed of its semi-axes.

261. If the axes of the ellipsoid (10) be taken as axes of co-ordinates, the equation assumes the form

$$I_1x^2 + I_2y^2 + I_3z^2 = M\epsilon^4, \tag{11}$$

where I_1, I_2, I_3 are the principal moments at the point O.

By Art. 259 we have $\rho^2 = \kappa^2/I = M\epsilon^4/I$; hence $I = M\epsilon^4/\rho^2$. If, therefore, equation (11) be divided by ρ^2, the following simple expression is obtained for finding the moment of inertia, I, for a line whose direction cosines referred to the principal axes are α, β, γ,

$$I = I_1\alpha^2 + I_2\beta^2 + I_3\gamma^2. \tag{12}$$

262. To make use of this form for I, the principal axes at the point O, *i.e.* the axes of the momental ellipsoid (10), must be known. The determination of the axes of an ellipsoid whose equation referred to the centre is given is a well-known problem of analytic geometry. It can be solved by considering that the semi-axes are those radii vectores of the surface that are normal to it. The direction cosines of the normal of any surface $F(x, y, z) = 0$ are proportional to the partial derivatives $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$. If, therefore, the radius vector ρ is a semi-axis, its direction-cosines α, β, γ must be proportional to the partial derivatives of (10); *i.e.* we must have

$$\frac{Ax - Fy - Ez}{\alpha} = \frac{-Fx + By - Dz}{\beta} = \frac{-Ex - Dy + Cz}{\gamma},$$

or dividing the numerators by ρ,

$$\frac{A\alpha - F\beta - E\gamma}{\alpha} = \frac{-F\alpha + B\beta - D\gamma}{\beta} = \frac{-E\alpha - D\beta + C\gamma}{\gamma}.$$

Denoting the common value of these fractions by I, we have

$$\alpha I = A\alpha - F\beta - E\gamma,\ \beta I = -F\alpha + B\beta - D\gamma,\ \gamma I = -E\alpha - D\beta + C\gamma;$$

multiplying these equations by α, β, γ, and adding, we find

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta,$$

which, compared with (9), shows that I is the moment of inertia for the axis (α, β, γ). To obtain it in function of A, B, C, D, E, F, we write the preceding three equations in the form

$$\begin{aligned} (I - A)\alpha + F\beta + E\gamma &= 0, \\ F\alpha + (I - B)\beta + D\gamma &= 0, \\ E\alpha + D\beta + (I - C)\gamma &= 0, \end{aligned} \qquad (13)$$

whence, eliminating α, β, γ, we find I determined by the cubic equation

$$\begin{vmatrix} I - A, & F, & E \\ F, & I - B, & D \\ E, & D, & I - C \end{vmatrix} = 0. \qquad (14)$$

The roots of this cubic are the three principal moments I_1, I_2, I_3 of the point O. The direction-cosines of the principal axes are then found by substituting successively I_1, I_2, I_3 in (13) and solving for α, β, γ.

263. The geometrical representation of the moments of inertia for all lines passing through a point by means of the radii vectores of the momental ellipsoid at the point, gives at once a number of propositions about these moments. It is only necessary to interpret properly the geometrical properties of the ellipsoid. Thus, it is known that the sum of the squares of the reciprocals of any three rectangular semi-diameters of an ellipsoid is constant. It follows that the sum of the three moments of inertia for any three rectangular lines passing through the same point has a constant value.

In general, the three principal moments of inertia I_1, I_2, I_3 at a point O are different. If, however, two of them are equal, say $I_2 = I_3$, the momental ellipsoid becomes an ellipsoid of revolution about the third, I_1, as axis; and it follows that the moments of inertia for all lines through O lying in the plane of the two equal axes are equal.

If $I_1 = I_2 = I_3$, the ellipsoid becomes a sphere, and the moments of inertia are the same for all lines passing through O.

264. If the equation of the momental ellipsoid at a point O be of the form $Ax^2 + By^2 + Cz^2 - 2Dyz = M\epsilon^4$; *i.e.* if the two conditions

$$E \equiv \Sigma mzx = 0, \quad F \equiv \Sigma mxy = 0$$

be fulfilled, the axis of x coincides with one of the three axes of the ellipsoid, the surface being symmetrical with respect to the yz-plane. Hence, *if the conditions* $E = 0$, $F = 0$ *are satisfied, the axis of* x *is a principal axis at the origin.* The converse is evidently also true; *i.e.* if a line is a principal axis at one of its points, then, taking this point as origin and the line as axis of x, the conditions $\Sigma mzx = 0$, $\Sigma mxy = 0$ must be satisfied.

It is easy to see that if a line be a principal axis at one of its points, say O, it will in general not be a principal axis at any other one of its points. For, taking the line as axis of x and O as origin, we have $\Sigma mzx = 0$, $\Sigma mxy = 0$. If now for a point O' on this line at the distance a from O the line is likewise a principal axis, the conditions

$$\Sigma mz(x-a) = 0, \quad \Sigma m(x-a)y = 0$$

must be fulfilled. These reduce to

$$\Sigma mz = 0,\quad \Sigma my = 0,$$

and show that the line must pass through the centroid. And as for a centroidal line these conditions are satisfied independently of the value of a, it appears that a centroidal principal axis is principal axis at every one of its points. Hence *a line cannot be principal axis at more than one of its points unless it pass through the centroid; in the latter case it is principal axis at every one of its points.*

265. All those lines passing through a given point O for which the moments of inertia have the same value I can be shown to form a cone of the second order whose principal diameters coincide with the axes of the momental ellipsoid at O. This cone is called an **equimomental cone.** Its equation is obtained by regarding I as constant in equation (12) and introducing rectangular co-ordinates. Multiplying (12) by $\alpha^2+\beta^2+\gamma^2=1$, we find

$$(I_1-I)\alpha^2+(I_2-I)\beta^2+(I_3-I)\gamma^2=0;$$

and multiplying by ρ^2, we obtain the equation of the equimomental cone in the form

$$(I_1-I)x^2+(I_2-I)y^2+(I_3-I)z^2=0. \tag{15}$$

266. A slightly different form of the equations (11), (12), (15) is often more convenient; it is obtained by introducing the three **principal radii of inertia** q_1, q_2, q_3 defined by the relations

$$I_1=Mq_1^2,\quad I_2=Mq_2^2,\quad I_3=Mq_3^2.$$

The equation (11) of the momental ellipsoid at the point O then assumes the form

$$q_1^2x^2+q_2^2y^2+q_3^2z^2=\epsilon^4. \tag{11'}$$

The expression of the radius of inertia q for any line (α, β, γ) through O becomes

$$q^2=q_1^2\alpha^2+q_2^2\beta^2+q_3^2\gamma^2. \tag{12'}$$

Dividing (11′) by the square of the radius vector, ρ^2, and comparing with (12′), we find

$$q=\frac{\epsilon^2}{\rho},\ \rho=\frac{\epsilon^2}{q}, \tag{16}$$

as is otherwise apparent from the fundamental property of the momental ellipsoid (Art. 259).

The equation of the sphere of radius q described about O as centre, $x^2+y^2+z^2=q^2$, together with (11′), represents the curve of intersection of the ellipsoid with the sphere. Through this sphero-conic passes the equimomental cone, all of whose lines have the moment of inertia $I=Mq^2$. Hence, the equation of this cone can be written in the form

$$\left(\frac{q_1^2}{\epsilon^4}-\frac{1}{q^2}\right)x^2+\left(\frac{q_2^2}{\epsilon^4}-\frac{1}{q^2}\right)y^2+\left(\frac{q_3^2}{\epsilon^4}-\frac{1}{q^2}\right)z^2=0. \tag{15′}$$

267. If we assume $I_1>I_2>I_3$, and hence $q_1>q_2>q_3$, q must be $<\epsilon^2/q_3$ and $>\epsilon^2/q_1$. As long as q is less than the middle semi-axis ϵ^2/q_2 of the ellipsoid, the axis of the cone coincides with the axis of z, but when $q>\epsilon^2/q_2$, the axis of x is the axis of the cone. For $q=\epsilon^2/q_2$ the cone degenerates into the pair of planes $(q_1^2-q_2^2)x^2-(q_2^2-q_3^2)z^2=0$. These are the planes of the central circular (or *cyclic*) sections of the ellipsoid; they divide the ellipsoid into four wedges, of which one pair contains all the equimomental cones whose axes coincide with the greatest axis of the ellipsoid, while the other pair contains all those whose axes lie along the least axis of the ellipsoid.

268. There is another ellipsoid closely connected with the theory of principal axes; it is obtained from the momental ellipsoid by the process of reciprocation.

About any point O (Fig. 32) taken as centre let us describe a sphere of radius ϵ, and construct for every point P its polar plane π with regard to the sphere. If P describe any surface, the plane π will envelop another surface which is called the *polar reciprocal* of the former surface with regard to the sphere.

Let Q be the intersection of OP with π, and put $OP=\rho$, $OQ=q$; then it appears from the figure that

$$\rho q=\epsilon^2. \tag{16}$$

269. It is easy to see that the polar reciprocal of the momental ellipsoid (11′) with respect to the sphere of radius ϵ is the ellipsoid

$$\frac{x^2}{q_1^2}+\frac{y^2}{q_2^2}+\frac{z^2}{q_3^2}=1. \tag{17}$$

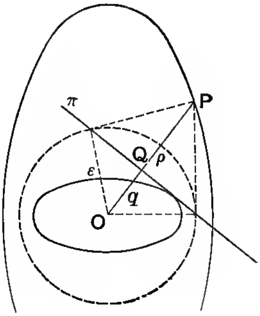

Fig. 32.

To prove this it is only necessary to show that the relation (16) is fulfilled for ρ as radius vector of (11′), and q as perpendicular to the tangent plane of (17). Now this tangent plane has the equation

$$\frac{x}{q_1^2}X+\frac{y}{q_2^2}Y+\frac{z}{q_3^2}Z=1;$$

hence we have for the direction cosines α, β, γ, and for the length q, of the perpendicular to the tangent plane

$$\frac{\alpha}{x/q_1^2}=\frac{\beta}{y/q_2^2}=\frac{\gamma}{z/q_3^2}=\frac{1}{[x^2/q_1^4+y^2/q_2^4+z^2/q_3^4]^{\frac{1}{2}}}=q.$$

These relations give $q_1\alpha=(x/q_1)q$, $q_2\beta=(y/q_2)q$, $q_3\gamma=(z/q_3)q$, whence

$$q_1^2\alpha^2+q_2^2\beta^2+q_3^2\gamma^2=\left(\frac{x^2}{q_1^2}+\frac{y^2}{q_2^2}+\frac{z^2}{q_3^2}\right)q^2=q^2. \tag{18}$$

For the radius vector ρ of (11′) whose direction cosines α, β, γ are the same as those of q we have by (11′):

$$\rho^2=\frac{\epsilon^4}{q_1^2\alpha^2+q_2^2\beta^2+q_3^2\gamma^2}.$$

Hence $\rho^2q^2=\epsilon^4$; and this is what we wished to prove.

270. The surface (17) has variously been called the *ellipsoid of gyration*, the *ellipsoid of inertia*, the **reciprocal ellipsoid.** We shall adopt the last name. The semi-axes of this ellipsoid are equal to the principal radii of inertia at the point O. The directions of its axes coincide with those of the momental ellipsoid; but the greatest axis of the former coincides with the least of the latter, and *vice versa.*

By comparing the equations (12′) and (18) it will be seen that q is the radius of inertia of the line (α, β, γ) on which it lies. Thus, *while the radius vector* $OP = \rho$ *of the momental ellipsoid is inversely proportional to the radius of inertia,* i.e. $\rho = \epsilon^2/q$, *the reciprocal ellipsoid gives the radius of inertia* q *for a line* l *as the segment cut off on this line by the perpendicular tangent plane.*

271. We are now prepared to determine the moment of inertia for any line in space. Let us construct at the centroid G of the given mass or body both the momental ellipsoid and its polar reciprocal. The former is usually called the **central ellipsoid** of the body; the latter we may call the **fundamental ellipsoid** of the body. As soon as this fundamental ellipsoid

$$\frac{x^2}{q_1^{\,2}} + \frac{y^2}{q_2^{\,2}} + \frac{z^2}{q_3^{\,2}} = 1$$

is known, the moment of inertia of the body for any line whatever can readily be found. For, by Art. 270, the radius of inertia q for any line l_0 passing through the centroid is equal to the segment OQ cut off on the line l_0 by the perpendicular tangent plane of the fundamental ellipsoid; and for any line l not passing through the centroid the square of the radius of inertia can be determined by first finding the square of the radius of inertia for the parallel centroidal line l_0 and then, by Art. 250, adding to it the square of the distance d of the centroid from the line l.

272. In the problem of determining the ellipsoids of inertia for a given body at any point, considerations of symmetry are of great assistance, similarly as in the problem of finding the centroid (compare Part II., Art. 47).

Suppose a given mass to have a plane of symmetry; then taking this plane as the yz-plane, and a perpendicular to it as the axis of x, there must be, for every particle of mass m, whose co-ordinates are x, y, z, another particle of equal mass m, whose co-ordinates are $-x, y, z$. It follows that the two products of inertia Σmzx and Σmxy both vanish, whatever the position of the other two co-ordinate planes. Hence any perpendicular to the plane of symmetry is a principal axis at its point of intersection with this plane.

If the mass have two planes of symmetry at right angles to each other, then taking one as yz-plane, the other as zx-plane, and hence their intersection as axis of x, it is evident that all three products of inertia vanish,

$$\Sigma myz = 0, \qquad \Sigma mzx = 0, \qquad \Sigma mxy = 0,$$

wherever the origin be taken on the intersection of the two planes. Hence, for any point on this intersection, the principal axes are the line of intersection of the two planes of symmetry, and the two perpendiculars to it, drawn in each plane.

If there be three planes of symmetry, their point of intersection is the centroid, and their lines of intersection are the principal axes at the centroid.

273. Exercises.

Determine the principal axes and radii at the centroid, the central and fundamental ellipsoids, and show how to find the moment of inertia for any line, in the following Exercises (1), (2), (3).

(1) Rectangular parallelepiped, the edges being $2a$, $2b$, $2c$. Find also the moments of inertia for the edges and diagonals, and specialize for the cube.

(2) Ellipsoid of semi-axes a, b, c. Determine also the radius of inertia for a parallel l to the shortest axis passing through the extremity of the longest axis.

(3) Right circular cone of height h and radius of base a. Find first the principal moments at the vertex; then transfer to the centroid.

(4) Determine the momental ellipsoid and the principal axes at a vertex of a cube whose edge is a.

(5) Determine the radius of inertia of a thin wire bent into a circle, for a line through the centre inclined at an angle α to the plane of the circle.

(6) A peg-top is composed of a cone of height H and radius a, and a hemispherical cap of the same radius. The point, to a distance h from the vertex of the cone, is made of a material three times as heavy as the rest. Find the moment of inertia for the axis of rotation; specialize for $h = a = \frac{1}{3} H$.

(7) Show that the principal axes at any point A, situated on one of the principal axes of a body, are parallel to the centroidal principal axes, and find their moments of inertia.

(8) For a given body of mass M find the points at which the momental ellipsoid reduces to a sphere.

(9) Determine a homogeneous ellipsoid having the same mass as a given body, and such that its moment of inertia for every line shall be the same as that of the given body.

3. DISTRIBUTION OF PRINCIPAL AXES IN SPACE.

274. It has been shown in the preceding articles how the principal axes can be determined at any particular point. The distribution of the principal axes throughout space and their position at the different points is brought out very graphically by means of the theory of confocal quadrics. It can be shown that the directions of the principal axes at any point are those of the principal diameters of the tangent cone drawn from this point as vertex to the fundamental ellipsoid; or, what amounts to the same thing, they are the directions of the normals of the three quadric surfaces passing through the point and confocal to the fundamental ellipsoid.

In order to explain and prove these propositions it will be necessary to give a short sketch of the theory of confocal conics and quadrics.

275. *Two conic sections are said to be* **confocal** *when they have the same foci.* The directions of the axes of all conics having the same two points S, S' as foci must evidently coincide, and the equation of such conics can be written in the form

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, \qquad (19)$$

where λ is an arbitrary parameter. For, whatever value may be assigned in this equation to λ, the distance of the centre O from either focus will always be $\sqrt{a^2+\lambda-(b^2+\lambda)}=\sqrt{a^2-b^2}$; it is therefore constant.

276. The individual curves of the whole system of confocal conics represented by (19) are obtained by giving to λ any particular value between $-\infty$ and $+\infty$; thus we may speak of the conic λ of the system.

For $\lambda=0$ we have the so-called fundamental conic $x^2/a^2+y^2/b^2=1$; this is an ellipse. To fix the ideas let us assume $a>b$. For all values of $\lambda>-b^2$, *i.e.* as long as $-b^2<\lambda<\infty$, the conics (19) are ellipses, beginning with the rectilinear segment SS' (which may be regarded as a degenerated ellipse $\lambda=-b^2$ whose minor axis is 0), expanding gradually, passing through the fundamental ellipse $\lambda=0$, and finally verging into a circle of infinite radius for $\lambda=\infty$.

It is thus geometrically evident that through every point in the plane will pass one, and only one, of these ellipses.

277. Let us next consider what the equation (19) represents when λ is algebraically less than $-b^2$. The values of λ that are $<-a^2$ give imaginary curves, and are of no importance for our purpose. But as long as $-a^2<\lambda<-b^2$, the curves are hyperbolas. The curve $\lambda=-b^2$ may now be regarded as a degenerated hyperbola collapsed into the two rays issuing in opposite directions from S and S' along the line SS'. The degenerated ellipse together with this degenerated hyperbola thus represents the whole axis of x.

As λ decreases, the hyperbola expands, and finally, for $\lambda=-a^2$, verges into the axis of y, which may be regarded as another degenerated hyperbola.

The system of confocal hyperbolas is thus seen to cover likewise the whole plane so that one, and only one, hyperbola of the system passes through every point of the plane.

278. The fact that every point of the plane has one ellipse and one hyperbola of the confocal system (19) passing through it allows us to regard the two values of the parameter λ that determine these two curves as co-ordinates of the point; they are called *elliptic co-ordinates*. If x, y be the rectangular Cartesian co-ordinates of the point, its elliptic co-ordinates λ_1, λ_2 are found as the roots of the equation (19) which is quadratic in λ. Conversely, to transform from elliptic to

Cartesian co-ordinates, that is, to express x and y in terms of λ_1 and λ_2, we have only to solve for x and y the two equations

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1, \quad \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1.$$

279. The two confocal conics that pass through the same point P intersect at right angles. For the tangent to the ellipse at P bisects the exterior angle at P in the triangle SPS', while the tangent to the hyperbola bisects the interior angle at the same point; in other words, the tangent to one curve is normal to the other, and *vice versa.* The elliptic system of co-ordinates is, therefore, an *orthogonal* system; the infinitesimal elements $d\lambda_1 \cdot d\lambda_2$ into which the two series of confocal conics (19) divide the plane are rectangular, though curvilinear.

280. These considerations are easily extended to space of three dimensions.

An ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ where } a>b>c,$$

has six real foci in its principal planes; two, S_1, S_1', in the xy-plane, on the axis of x, at a distance $OS_1=\sqrt{a^2-b^2}$ from the centre O; two, S_2, S_2', in the yz-plane, on the axis of y, at the distance $OS_2=\sqrt{b^2-c^2}$ from the centre; and two, S_3, S_3', in the zx-plane, on the axis of x, at the distance $OS_3=\sqrt{a^2-c^2}$ from the centre. It should be noticed that, since $b>c$, we have $OS_3>OS_1$; *i.e.* S_1, S_1' lie between S_3, S_3' on the axis of x.

The same holds for hyperboloids.

281. *Two quadric surfaces are said to be confocal when their principal sections are confocal conics.* Now this will be the case for two quadric surfaces whose semi-axes are a_1, b_1, c_1, and a_2, b_2, c_2, if the directions of their axes coincide and if

$$a_1^2-b_1^2=a_2^2-b_2^2, \quad b_1^2-c_1^2=b_2^2-c_2^2, \quad a_1^2-c_1^2=a_2^2-c_2^2.$$

Writing these conditions in the form

$$a_2^2-a_1^2=b_2^2-b_1^2=c_2^2-c_1^2, \text{ say } =\lambda,$$

we find $a_2^2=a_1^2+\lambda$, $b_2^2=b_1^2+\lambda$, $c_2^2=c_1^2+\lambda$. Hence the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \tag{20}$$

where λ is a variable parameter, represents a system of confocal quadric surfaces.

282. As long as λ is algebraically greater than $-c^2$, the equation (20) represents ellipsoids. For $\lambda = -c^2$ the surface collapses into the interior area of the ellipse in the xy-plane whose vertices are the foci S_2, S_2' and S_3, S_3'. For as λ approaches the limit $-c^2$, the three semi-axes of (20) approach the limits $\sqrt{a^2-c^2}$, $\sqrt{b^2-c^2}$, 0, respectively. This limiting ellipse is called the *focal ellipse*. Its foci are the points S_1, S_1', since $a^2 - c^2 - (b^2 - c^2) = a^2 - b^2$.

When λ is algebraically $< -c^2$, but $> -a^2$, the equation (20) represents hyperboloids; for values of $\lambda < -a^2$ it is not satisfied by any real points. As long as $-b^2 < \lambda < -c^2$, the surfaces are hyperboloids of one sheet. The limiting surface $\lambda = -c^2$ now represents the exterior area of the focal ellipse in the xy-plane. The limiting hyperboloid of one sheet for $\lambda = -b^2$ is the area in the zx-plane bounded by the hyperbola whose vertices are S_1, S_1', and whose foci are S_3, S_3'. This is called the *focal hyperbola*.

Finally, when $-a^2 < \lambda < -b^2$, the surfaces are hyperboloids of two sheets, the limiting hyperboloid $\lambda = -a^2$ collapsing into the yz-plane.

283. It appears from these geometrical considerations, that there are passing through every point of space three surfaces confocal to the fundamental ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ and to each other, viz.: an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. This can also be shown analytically, as there is no difficulty in proving that the equation (20) has three real roots, say λ_1, λ_2, λ_3, for every set of real values of x, y, z, and that these roots are confined between such limits as to give the three surfaces just mentioned.

The quantities λ_1, λ_2, λ_3 can therefore be taken as co-ordinates of the point (x, y, z); and these *elliptic co-ordinates* of the point are, geometrically, the parameters of the three quadric surfaces passing through the point and confocal to the fundamental ellipsoid; while, analytically, they are the three roots of the cubic (20). To express x, y, z in terms of the elliptic co-ordinates, it is only necessary to solve for x, y, z the three equations obtained by substituting in (20) successively λ_1, λ_2, λ_3 for λ.

284. The geometrical meaning of the parameter λ will appear by considering two parallel tangent planes π_0 and π_λ (on the same side of the origin), the former (π_0) tangent to the fundamental ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, the latter ($\pi_\lambda$) tangent to any confocal surface λ or $x^2/(a^2+\lambda)+y^2/(b^2+\lambda)+z^2/(c^2+\lambda)=1$. The perpendiculars q_0, q_λ, let fall from the origin O on these tangent planes π_0, π_λ, are given by the relations (the proof being the same as in Art. 269).

$$q_0^2=a^2\alpha^2+b^2\beta^2+c^2\gamma^2, \tag{21}$$

$$q_\lambda^2=(a^2+\lambda)\alpha^2+(b^2+\lambda)\beta^2+(c^2+\lambda)\gamma^2, \tag{22}$$

where α, β, γ are the direction-cosines of the common normal of the planes π_0, π_λ. Subtracting (21) from (22), we find, since $\alpha^2+\beta^2+\gamma^2=1$,

$$q_\lambda^2-q_0^2=\lambda; \tag{23}$$

i.e. *the parameter λ of any one of the confocal surfaces* (20) *is equal to the difference of the squares of the perpendiculars let fall from the common centre on any tangent plane to the surface λ, and on the parallel tangent plane to the fundamental ellipsoid $\lambda=0$.*

285. Let us now apply these results to the question of the distribution of the principal axes throughout space.

We take the centroid G of the given body as origin, and select as fundamental ellipsoid of our confocal system the polar reciprocal of the central ellipsoid, *i.e.* the ellipsoid (17) formed for the centroid, for which the name "fundamental ellipsoid of the body" was introduced in Art. 271. Its equation is

$$\frac{x^2}{q_1^2}+\frac{y^2}{q_2^2}+\frac{z^2}{q_3^2}=1,$$

if q_1, q_2, q_3 are the principal radii of inertia of the body.

The radius of inertia q_0 for any centroidal line l_0 can be constructed (Art. 270) by laying a tangent plane to this ellipsoid perpendicular to the line l_0; if this line meets the tangent plane in Q_0 (Fig. 33), then $q_0=GQ_0$. Analytically, if α, β, γ be the direction-cosines of l_0, q_0 is given by formula (21) or (12′).

286. To find the radius of inertia q for a line l, parallel to l_0, and passing through any point P, we lay through P a plane π_λ, perpendicular to l, and a parallel plane π_0, tangent to the fundamental ellipsoid; let Q_λ, Q_0 be the intersections of these planes with the centroidal line l_0. Then, putting $GQ_0=q_0$, $GQ_\lambda=q_\lambda$, $GP=r$, $PQ_\lambda=d$, we have, by Art. 250,

$$q^2=q_0^2+d^2.$$

The figure gives the relation $d^2 = r^2 - q_\lambda^2$, which, in combination with (23), reduces the expression for the radius of inertia for the line l to the simple form:

$$q^2 = r^2 - \lambda. \tag{24}$$

287. The value of $r^2 - \lambda$, and hence the value of q, remains the same for the perpendiculars to all planes through P, tangent to the same quadric surface λ: these perpendiculars form, therefore, an equimomental cone at P. By varying λ we thus obtain all the equimomental cones at P. The principal diameters of all these cones coincide in direction, since they coincide with the directions of the principal axes of the momental ellipsoid at P (see Art. 265); but they also coincide with the principal diameters of the cones enveloped by the tangent planes π_λ. It thus appears that *the principal axes at the point* P *coincide in direction with the principal diameters of the tangent cone from* P *as vertex to the fundamental ellipsoid* $x^2/q_1^2 + y^2/q_2^2 + z^2/q_3^2 = 1$.

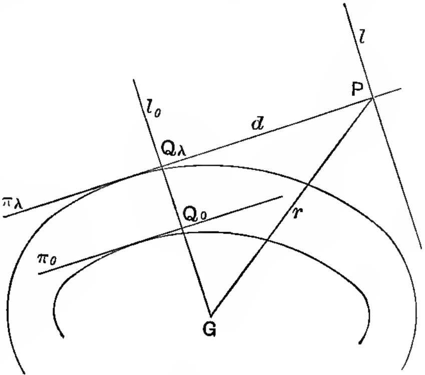

Fig. 33.

288. Instead of the fundamental ellipsoid, we might have used any quadric surface λ confocal to it. In particular, we may select the confocal surfaces λ_1, λ_2, λ_3 that pass through P. For each of these the cone of the tangent planes collapses into a plane, viz. the tangent plane to the surface at P, while the cone of the perpendiculars reduces to a single line, viz. the normal to the surface at P. Thus we find that *the principal axes at any point* P *coincide in direction with the normals to the three quadric surfaces, confocal to the fundamental ellipsoid and passing through* P.

For the magnitudes of the principal radii q_x, q_y, q_z at P, we evidently have

$$q_x^2 = r^2 - \lambda_1, \qquad q_y^2 = r^2 - \lambda_2, \qquad q_z^2 = r^2 - \lambda_3.$$

289. Exercise.

(1) The principal radii q_1, q_2, q_3 of a body being given, find the equation of the momental ellipsoid at any point P, referred to axes

through this point P parallel to the principal axes of the body; determine the directions of the principal axes at P, and show that these directions coincide with the normals of the three surfaces passing through P and confocal to the fundamental ellipsoid of the body.

290. A brief account of the theory of moments of inertia will be found in B. Williamson, *Integral Calculus*, 6th ed., London, Longmans, 1891, pp. 291–312. The subject is discussed very fully in E. J. Routh, *Dynamics of a system of rigid bodies*, Part I., 5th ed., London, Macmillan, 1891, pp. 1–49; and in B. Price, *Analytical mechanics*, Vol. II., 2d ed., Oxford, Clarendon Press, 1889, Chapter IV. The student will also consult with advantage W. Schell, *Theorie der Bewegung und der Kräfte*, Vol. I., 2d ed., Leipzig, Teubner, 1879, pp. 100–143, and A. Cayley, *Report on the progress of the solution of certain special problems of dynamics*, in the Report of the 32d meeting of the British Association, for 1862, London, Murray, 1863, pp. 223–229.

III. *Rigid Body with a Fixed Axis.*

291. A rigid body with a fixed axis has but one degree of freedom. Its motion is fully determined by the motion of any one of its points (not situated on the axis), and any such point must move in a circle about the axis. Any particular position of the body is, therefore, determined by a single variable, or co-ordinate, such as the angle of rotation. Just as the equilibrium of such a body depends on a single condition (see Part II., Art. 227), so its motion is given by a single equation.

292. The equation of motion can be derived directly from the proposition of angular momentum (Art. 224). Let r be the distance of any particle m of the body from the fixed axis, ω the angular velocity at the time t; then $m\omega r$ is the momentum of the particle, and $m\omega r^2$ its moment, or the angular momentum of the particle, about the axis. At any given instant t, ω has the same value for all particles. Hence, the angular momentum of the body is $\omega \Sigma mr^2 = \omega I$, where $I = \Sigma mr^2$ is the moment of inertia of the body for the fixed axis.

Now, by Art. 224, the rate at which the angular momentum of the body about the axis changes with the time is equal to the sum of the moments of all the external forces about the same axis. Denoting this resulting moment by H, and considering that the moment of inertia for the fixed axis is independent of the time, we have the **equation of motion**

$$\frac{d\omega}{dt} = \frac{H}{I}; \qquad (1)$$

i.e. *the angular acceleration about the fixed axis is equal to the moment of all the external forces about this axis, divided by the moment of inertia of the body for the same axis.*

293. The same result can of course be obtained from any one of the equations (6) or (7), Art. 224. Thus, taking the fixed line as axis of z, the third of the equation (7), viz.

$$\frac{d}{dt}\Sigma m(x\dot{y} - y\dot{x}) = H_z,$$

must be used. Now, for rotation of angular velocity ω about the axis of z, we have $\dot{x}=-\omega y$, $\dot{y}=\omega x$. Hence

$$\Sigma m(x\dot{y}-y\dot{x})=\omega\Sigma m(x^2+y^2)=\omega\Sigma mr^2=\omega I.$$

The equation assumes, therefore, the form (1).

294. The reactions of the fixed axis do not enter into the composition of the resulting moment H. As they intersect the axis, their moments about this axis are zero.

The student should notice the close analogy between equation (1) and the equation for the rectilinear motion of a particle,

$$\frac{dv}{dt}=\frac{F}{m},$$

where v is the velocity and F the resultant of all the forces acting on the particle.

The expression for the *kinetic energy* of a body rotating about a fixed axis is

$$T=\Sigma\tfrac{1}{2}mv^2=\Sigma\tfrac{1}{2}m\omega^2r^2=\tfrac{1}{2}I\omega^2, \tag{2}$$

and has also a form similar to that for the kinetic energy of a particle m moving with velocity v in a straight line, viz.

$$T=\tfrac{1}{2}mv^2.$$

295. Let us denote the angle of rotation by θ, so that $\omega=d\theta/dt$, $d\omega/dt=d^2\theta/dt^2$. If the resulting moment be constant or a given function of θ, say $H=f(\theta)$, the equation of motion

$$I\frac{d^2\theta}{dt^2}=f(\theta)$$

can be integrated once, and gives

$$\tfrac{1}{2}I(\omega^2-\omega_0^2)=\int_{\theta_0}^{\theta}f(\theta)d\theta, \tag{3}$$

where ω_0 is the angular velocity corresponding to the angle θ_0.

This is the *equation of kinetic energy*. It might have been derived directly, according to Art. 234, by expressing that the increase of the kinetic energy equals the work of the forces.

The kinetic energy is given by (2). The work of a force F in a plane perpendicular to the axis, at the distance p from the axis, is $F \cdot p d\theta$ for an infinitesimal rotation of angle $d\theta$; hence, the sum of the elementary works of all the forces $= \Sigma F p d\theta = H d\theta = f(\theta) d\theta$.

296. While thus the motion of a rigid body about a fixed axis is given by a single equation, the other equations of motion of a rigid body are required to determine the *reactions of the fixed axis* (comp. Part II., Art. 227).

The axis will be fixed if any two of its points A, B are fixed. The reaction of the fixed point A can be resolved into three components A_x, A_y, A_z, that of B into B_x, B_y, B_z. By introducing these reactions the body becomes free; and the system composed of these reactions, of the external forces, and of the reversed effective forces must be in equilibrium. We take again the axis of rotation as axis of z (Fig. 34) so that the z-co-ordinates of the particles are constant, and hence $\dot z = 0$, $\ddot z = 0$; and we put $OA = a$, $OB = b$. Then the six equations of motion are (see Art. 223 (4) and Art. 224 (6)):

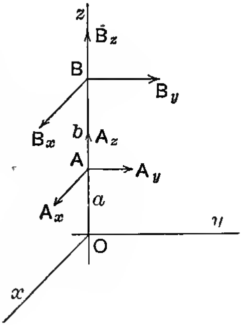

Fig. 34.

$$\begin{aligned}
\Sigma m\ddot x &= \Sigma X + A_x + B_x,\\
\Sigma m\ddot y &= \Sigma Y + A_y + B_y,\\
0 &= \Sigma Z + A_z + B_z,\\
-\Sigma m z\ddot y &= \Sigma(yZ - zY) - aA_y - bB_y,\\
\Sigma m z\ddot x &= \Sigma(zX - xZ) + aA_x + bB_x,\\
\Sigma m(x\ddot y - y\ddot x) &= \Sigma(xY - yX).
\end{aligned}$$

297. It remains to introduce into these equations the values for $\ddot x$, $\ddot y$. As the motion is a pure rotation, we have (see Part I., Art. 245) $\dot x = -\omega y$, $\dot y = \omega x$; hence, $\ddot x = -\dot\omega y - \omega^2 x$, $\ddot y = \dot\omega x - \omega^2 y$. Summing over the whole body, we find

$$\begin{aligned}
\Sigma m\ddot x &= -\dot\omega \Sigma my - \omega^2 \Sigma mx = -M\dot\omega\bar y - M\omega^2\bar x,\\
\Sigma m\ddot y &= \quad \dot\omega \Sigma mx - \omega^2 \Sigma my = \quad M\dot\omega\bar x - M\omega^2\bar y,
\end{aligned}$$

where $\bar{x}, \bar{y}$ are the co-ordinates of the centroid; and

$$\begin{aligned}
-\Sigma mz\ddot{y} &= -\dot{\omega}\Sigma mzx + \omega^2\Sigma myz = -E\dot{\omega} + D\omega^2,\\
\Sigma mz\ddot{x} &= -\dot{\omega}\Sigma myz - \omega^2\Sigma mzx = -D\dot{\omega} - E\omega^2,\\
\Sigma m(x\ddot{y} - y\ddot{x}) &= \dot{\omega}\Sigma mx^2 - \omega^2\Sigma mxy + \dot{\omega}\Sigma my^2 + \omega^2\Sigma mxy = C\dot{\omega},
\end{aligned}$$

where $C = \Sigma m(x^2 + y^2)$, $D = \Sigma myz$, $E = \Sigma mzx$ are the notations introduced in Art. 255.

With these values the equations of motion assume the form:

$$\begin{aligned}
-M\bar{x}\omega^2 - M\bar{y}\dot{\omega} &= \Sigma X + A_x + B_x,\\
-M\bar{y}\omega^2 + M\bar{x}\dot{\omega} &= \Sigma Y + A_y + B_y,\\
0 &= \Sigma Z + A_z + B_z,\\
D\omega^2 - E\dot{\omega} &= \Sigma(yZ - zY) - aA_y - bB_y,\\
-E\omega^2 - D\dot{\omega} &= \Sigma(zX - xZ) + aA_x + bB_x,\\
C\dot{\omega} &= \Sigma(xY - yX).
\end{aligned} \tag{4}$$

298. The last equation is identical with equation (1).

The components of the reactions along the axis of rotation occur only in the third equation, and can therefore not be found separately. The longitudinal pressure on the axis is

$$= -A_z - B_z = \Sigma Z.$$

The remaining four equations are sufficient to determine A_x, A_y, B_x, B_y.

The total stress to which the axis is subject, instead of being resolved into two forces, at A and B, can be reduced for the origin O to a force and a couple (see Fig. 34). The equations (4) give for the components of the force

$$\begin{aligned}
-A_x - B_x &= \Sigma X + M\bar{x}\omega^2 + M\bar{y}\dot{\omega},\\
-A_y - B_y &= \Sigma Y + M\bar{y}\omega^2 - M\bar{x}\dot{\omega},\\
-A_z - B_z &= \Sigma Z.
\end{aligned} \tag{5}$$

This force consists of the resultant of the external forces,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2},$$

and two forces in the xy-plane which form the reversed effective force of the centroid; for $M\bar{x}\omega^2$ and $M\bar{y}\omega^2$ give as resultant the centrifugal force $M\omega^2\sqrt{\bar{x}^2+\bar{y}^2}=M\omega^2\bar{r}$, directed from the axis towards the projection of the centroid on the xy-plane, while $M\bar{y}\dot{\omega}, -M\bar{x}\omega$ form the tangential resultant $M\omega\bar{r}$, perpendicular to the plane through axis and centroid.

The couple has a component in the yz-plane, and one in the zx-plane, viz.:

$$\begin{aligned} aA_y+bB_y &= \Sigma(yZ-zY)-D\omega^2+E\dot{\omega}, \\ -aA_x-bB_x &= \Sigma(zX-xZ)+E\omega^2+D\dot{\omega}, \end{aligned} \qquad (6)$$

while the component in the xy-plane is zero. The resultant couple lies, therefore, in a plane passing through the axis of rotation.

299. In the particular case *when no forces* X, Y, Z *are acting on the body*, the last of the equations (4), or equation (1), shows that *the angular velocity* ω *remains constant*. The stress on the axis of rotation will, however, exist; and the axis will in general tend to change both its direction, owing to the couple (6), and its position, owing to the force (5).

If the axis be not fixed as a whole, but only one of its points, the origin, be fixed, the force (5) is taken up by the fixed point, while the couple (6) will change the direction of the axis. Now this couple vanishes if, in addition to the absence of external forces, the conditions

$$D\equiv\Sigma myz=0, \quad E\equiv\Sigma mzx=0 \qquad (7)$$

are fulfilled. In this case the body would continue to rotate about the axis of z even if this axis were not fixed; provided that the origin is a fixed point. A line having this property is called a **permanent axis of rotation.**

As the meaning of the conditions (7) is that the axis of z is a principal axis of inertia at the origin (see Art. 264), we have the proposition that *if a rigid body with a fixed point, not acted upon by any forces, begin to rotate about one of the principal axes at this point, it will continue to rotate uniformly about the same axis.* In other words, the principal axes at any point are always, and are the only, permanent axes of rotation. This can be regarded as the dynamical definition of principal axes.

300. It appears from the equations (5) that the *position* of the axis of rotation will remain the same if, in addition to the absence of external forces, the conditions

$$\bar{x}=0,\ \bar{y}=0 \tag{8}$$

be fulfilled; for in this case the components of the force (5) all vanish. If, moreover, the axis of rotation be a principal axis, the rotation will continue to take place about the same line even when the body has no fixed point.

The conditions (8) mean that the centroid lies on the axis of z; and it is known (Art. 264) that a centroidal principal axis is a principal axis at every one of its points. The axis of z must therefore be a principal axis of the body, *i.e.* a principal axis at the centroid. We have, therefore, the proposition: *If a free rigid body, not acted upon by any forces, begin to rotate about one of its centroidal principal axes, it will continue to rotate uniformly about the same line.*

301. A rigid body with a fixed *horizontal* axis is called a **compound pendulum** if the only external force acting is the weight of the body.

The plane through axis and centroid will make, with the vertical plane (downwards) through the axis, an angle θ, which we may take as angle of rotation, so that $\omega=d\theta/dt$ (Fig. 35). The weights of the particles, being all parallel and proportional to their masses, have a single resultant Mg passing through the centroid G. Hence, if h be the perpendicular distance OG of the centroid from the axis, the moment of the external forces is $H=-Mgh\sin\theta$; and if the radius of inertia of the body for the centroidal axis parallel to the axis of rotation be q, the moment of inertia for the latter axis is $I=M(q^2+h^2)$.

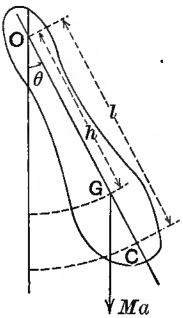

Fig. 35.

With these values the equation of motion (1) assumes the simple form

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{q^2+h^2}\sin\theta. \tag{9}$$

As shown in Art. 175, the equation of the *simple* pendulum of length l is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

The two equations differ only in the constant factor of $\sin\theta$, and it appears that *the motion of a compound pendulum is the same as that of a simple pendulum whose length is*

$$l = h + \frac{q^2}{h}. \tag{10}$$

302. The problem of the compound pendulum has thus been reduced to that of the simple pendulum. The length l is called *the length of the equivalent simple pendulum.* The foot O (Fig. 35) of the perpendicular let fall from the centroid on the axis is called *the centre of suspension.* If on the line OG a length $OC=l$ be laid off, the point C is called *the centre of oscillation.* It appears, from (10), that G lies between O and C.

The relation (10) can be written in the form

$$h(l-h) = q^2, \text{ or } OG \cdot GC = \text{const.}$$

As this relation is not altered by interchanging O and C, it follows that *the centres of oscillation and suspension are interchangeable; i.e.* the period of a compound pendulum remains the same if it be made to swing about a parallel axis through the centre of oscillation.

303. Exercises.

(1) A pendulum, formed of a cylindrical rod of radius a and length L, swings about a diameter of one of the bases. Find the time of a small oscillation.

(2) A cube, whose edge is a, swings as a pendulum about an edge. Find the length of the equivalent simple pendulum.

(3) A circular disc of radius r revolves uniformly about its axis, making 100 revolutions per minute. What is its kinetic energy?

(4) A fly-wheel of radius r, in which a mass, equal to that of the disc in Ex. (3), is distributed uniformly along the rim, has the same angular velocity as the disc. Neglecting the mass of the nave and spokes, determine its kinetic energy, and compare it with that of the disc.

(5) A fly-wheel of 12 ft. diameter, whose rim weighs 12 tons, makes 50 revolutions per minute. Find its kinetic energy in foot-pounds.

(6) A fly-wheel of radius r and mass m is making N revolutions per minute when the steam is shut off. If the radius of the shaft be r', and the coefficient of friction μ, find after how many revolutions the wheel will come to rest owing to the axle friction.

(7) A fly-wheel of 10 ft. diameter, weighing 5 tons, is making 40 revolutions when thrown out of gear. In what time does it come to rest if the diameter of the axle is 6 in. and the coefficient of friction $\mu = 0.05$?

(8) A uniform straight rod of length l is hinged at one end so as to turn freely in a vertical plane. If it be dropped from a horizontal position, with what angular velocity does it pass through the vertical position? (Equate the kinetic energy to the work of gravity.)

304. Impulses. Suppose a rigid body with a fixed axis is acted upon, when at rest, by a single impulse F, in a plane perpendicular to the axis and at the distance p from the axis. It is required to determine the initial motion of the body just after impact.

As the impulsive reactions of the fixed axis have no moment about this axis, the initial angular momentum of the body about the fixed axis must be equal to the moment of the impulse F about the same axis; *i.e.* to Fp. If ω is the initial angular velocity, the momentum of a particle m at the distance r from the axis is $m\omega r$; hence the angular momentum of the body $= \Sigma m\omega r^2 = \omega \Sigma m r^2 = \omega I$, where I is the moment of inertia of the body for the fixed axis. Hence we have

$$\omega = \frac{Fp}{I}. \tag{11}$$

305. Let the impulse F be produced by the inelastic impact of a particle of mass m moving with a velocity u. It would not be correct to put $F=mu$ in (11); as the particle after impact continues to move with the body with a certain velocity v, it does not actually give up to the body its whole momentum, but only the amount $F=m(u-v)$, provided that u and v have the same direction. With this assumption, which evidently means that the particle meets the body at some point of the plane passing through the axis and perpendicular to u, the velocity after impact is $v=\omega p$. With the value (11) of ω this gives

$$F=m(u-v)=mu-\frac{mp^2}{I}F,$$

whence $F=muI/(I+mp^2)$, and finally, by (11),

$$\omega=\frac{mup}{I+mp^2}. \tag{12}$$

As mp^2 is the moment of inertia of the particle for the fixed axis, this formula shows that we may substitute in (11) the whole momentum mu for F if we increase the moment of inertia of the body by that of the particle; in other words, that the particle may be regarded as giving up its whole momentum if it be taken into account that after impact it forms part of the body.

306. It is easy to see how the considerations of the last two articles can be generalized. When any number of impulses act in various directions on a rigid body with a fixed axis, the initial angular velocity will be determined by

$$\omega=\frac{H}{I}, \tag{13}$$

where H is the sum of the moments of all the impulses about the fixed axis.

307. To determine the impulsive stress produced on the axis by a single impulse F, let us write out the general equations of the impulsive motion.

Take the fixed axis as the axis of z and the zx-plane through the centroid G (Fig. 36), and let $\bar{x}$, 0, 0 be the co-ordinates of G, and x_1, y_1, z_1 those of the point of application P of the impulse F. The components of F may be denoted by X, Y, Z; those of the reactions of the axis by A_x, A_y, A_z, B_x, B_y, B_z, similarly as in Art. 296.

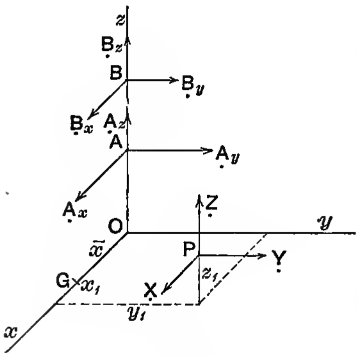

Fig. 36.

As the initial motion after impact is a rotation about the axis of z, we have $\dot{x} = -\omega y$, $\dot{y} = \omega x$, $\dot{z} = 0$, so that the momentum of a particle of mass m has the components $-m\omega y$, $m\omega x$, 0. Reducing these momenta to the origin O, we find a resultant momentum whose components are $-\omega \Sigma my = 0$, $\omega \Sigma mx = M\omega\bar{x}$, 0; and a resulting couple whose vector has the components $-\omega \Sigma mzx = -E\omega$, $-\omega \Sigma myz = -D\omega$, $\omega \Sigma m(x^2 + y^2) = C\omega$, where C, D, E have the same meaning as in Art. 297.

The six equations of motion just after the impulse are therefore, if the body was originally at rest:

$$\begin{aligned} 0 &= X + A_x + B_x, \\ M\bar{x}\omega &= Y + A_y + B_y, \\ 0 &= Z + A_z + B_z, \\ -E\omega &= y_1 Z - z_1 Y - aA_y - bB_y, \\ -D\omega &= z_1 X - x_1 Z + aA_x + bB_x, \\ C\omega &= x_1 Y - y_1 X. \end{aligned} \qquad (14)$$

308. The last of these equations is nothing but the equation (11). The components A_z, B_z along the axis cannot be determined separately; the other components of the reactions can be found from the first, second, fourth, and fifth equations.

The impulsive stress to which the axis is subjected by the impulse, or the so-called *percussion of the axis*, instead of being represented by two impulses A, B as above, can also be regarded as composed of an impulse whose components are

$$-\underset{.}{A}_x-\underset{.}{B}_x=\underset{.}{X},\quad -\underset{.}{A}_y-\underset{.}{B}_y=\underset{.}{Y}-M\bar{x}\omega,\quad -\underset{.}{A}_z-\underset{.}{B}_z=\underset{.}{Z},$$

and an impulsive couple whose vector has the components

$$a\underset{.}{A}_y+b\underset{.}{B}_y=y_1\underset{.}{Z}-z_1\underset{.}{Y}+E\omega,\quad -a\underset{.}{A}_x-b\underset{.}{B}_x=z_1\underset{.}{X}-x_1\underset{.}{Z}+D\omega,\ 0.$$

The last component being zero, the resulting couple lies in a plane passing through the axis of z.

If there were any number of impulses acting on the body simultaneously, the effect on the axis could be determined in the same way, except that the quantities $\underset{.}{X}$, $\underset{.}{Y}$, $\underset{.}{Z}$, $y_1\underset{.}{Z}-z_1\underset{.}{Y}$, $z_1\underset{.}{X}-x_1\underset{.}{Z}$, must be replaced by the corresponding sums.

309. It follows from the preceding article that the conditions under which a single impulse acting on a rigid body with a fixed axis will produce no stress on the axis are

$$\underset{.}{X}=0,\quad \underset{.}{Y}=M\bar{x}\omega,\quad \underset{.}{Z}=0,\quad -z_1M\bar{x}+E=0,\quad D=0. \qquad (15)$$

If these conditions are fulfilled, the resulting motion will be the same even when the axis is free.

The first and third equations show that *the impulse must be perpendicular to the plane passing through axis and centroid.* The meaning of the fourth and fifth conditions becomes apparent if the xy-plane be taken so as to pass through the point of application P of the impulse. The new origin O' is the foot of the perpendicular let fall from P on the fixed axis. To transform the conditions (15) to the new system it is only necessary to substitute $z+z_1$ for z; the first three conditions are not affected, and the last two become

$$-z_1M\bar{x}+\Sigma mzx+z_1\Sigma mx=0,\quad \Sigma myz+z_1\Sigma my=0,$$

or, since $\Sigma mx = M\bar{x}$, $\Sigma my = 0$,

$$E' = 0, \quad D' = 0,$$

where E', D' are the products of inertia at O'.

It thus appears that *the axis of* z *must be a principal axis at the foot of the perpendicular let fall on this axis from the point of application of the impulse.*

310. It should be noticed that a line taken at random in a body is not necessarily a principal axis at any one of its points. But if a line is a principal axis at a point O', then it is always possible to determine an impulse that will produce no stress on this line so that the body will begin to rotate about it as axis even though it be not fixed. As shown in the last article, the impulse must be $= M\bar{x}\omega$, and must be directed at right angles to the plane through axis and centroid. The point where it meets this plane is called the **centre of percussion.** Its distance x_1 from the axis is found from the equation of motion, viz. the last of the equations (14) which, owing to the conditions (15), reduces to

$$C = M\bar{x}x_1.$$

If q' be the radius of inertia of the body for a parallel centroidal axis, we have $C = M(q'^2 + \bar{x}^2)$; hence

$$x_1 = \bar{x} + \frac{q'^2}{\bar{x}}. \tag{16}$$

Hence, if a given line l be principal axis for one of its points O', there exists a centre of percussion; it lies on the intersection of the plane (l, G) with the plane through O' perpendicular to l, at the distance x_1, given by (16), from the line l. An impulse $M\bar{x}\omega$ through the centre of percussion at right angles to the plane through axis and centroid, while producing no percussion on the axis, sets the body rotating with angular velocity ω if it was originally at rest; on the other hand, if the body was originally in rotation about the axis, such an impulse can bring the body to rest without affecting the axis.

IV. *Rigid Body with a Fixed Point.*

311. A rigid body with a fixed point has three degrees of freedom. Any one of its points, with the exception of the fixed point O, is constrained to the surface of a sphere and has therefore two degrees of freedom; and the body itself can turn about the line joining this point to O. The motion consists, at any instant, of an infinitesimal rotation about an axis passing through O (see Part I., Arts. 32–35). Both the angular velocity and the direction of the instantaneous axis vary in the course of time.

312. We begin with the study of the *instantaneous motion* of the body, which may be regarded as due to the action of a system of impulses on the body at rest. This will lead to the solution of the converse problem, viz. *to determine the initial motion produced by a given system of impulses.*

I. INITIAL MOTION DUE TO IMPULSES.

313. The body rotates at the time t with angular velocity ω about the instantaneous axis l which passes through the fixed point O. It is required to determine a system of impulses that would produce this motion if acting on the body at rest.

For O as origin, let $\underset{\cdot}{R}$ be the resultant and $\underset{\cdot}{H}$ the resulting couple of these impulses. If the impulsive reaction $\underset{\cdot}{A}$ of the fixed point O be combined with them, the body can be regarded as free, and its instantaneous motion is determined by the equations (19) and (20), Art. 238. It is only necessary, in the equations (19), to add to the components $\underset{\cdot}{R}_x$, $\underset{\cdot}{R}_y$, $\underset{\cdot}{R}_z$ of $\underset{\cdot}{R}$ those of $\underset{\cdot}{A}$, while the right-hand members of (20) are not affected by $\underset{\cdot}{A}$, since its moment is zero for every axis through O.

314. It remains to form the sums in the left-hand members of (19) and (20) for our case; *i.e.* to reduce the system of momenta $m\dot{x}$, $m\dot{y}$, $m\dot{z}$ of the particles to its resultant and resultant couple for a fixed rectangular system of axes through O.

The resultant momentum has evidently the components

$$\Sigma m\dot{x}=M\dot{\bar{x}},\quad \Sigma m\dot{y}=M\dot{\bar{y}},\quad \Sigma m\dot{z}=M\dot{\bar{z}},$$

where $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ are the components of the velocity of the centroid at the time t, and M is the mass of the body. Hence the equations (19) become

$$M\dot{\bar{x}}=\underset{\cdot}{R}_x+\underset{\cdot}{A}_x,\quad M\dot{\bar{y}}=\underset{\cdot}{R}_y+\underset{\cdot}{A}_y,\quad M\dot{\bar{z}}=\underset{\cdot}{R}_z+\underset{\cdot}{A}_z. \tag{1}$$

These equations serve to determine the impulsive pressure $-\underset{\cdot}{A}=-\sqrt{A_x^2+A_y^2+A^2}$ on the fixed point O in magnitude and direction.

315. To form the moment $\Sigma m(y\dot{z}-z\dot{y})$ of the momenta of the particles, *i.e.* the angular momentum of the body, about the axis of x, we resolve the angular velocity ω into its components ω_x, ω_y, ω_z along the axes and observe that the components of the linear velocity of any point (x, y, z) arising from the rotation are (Part I., Art. 293):

$$\dot{x}=\omega_y z-\omega_z y,\quad \dot{y}=\omega_z x-\omega_x z,\quad \dot{z}=\omega_x y-\omega_y x.$$

Substituting these values, we find

$$\Sigma m\,(y\dot{z}-z\dot{y})=\omega_x\Sigma my^2-\omega_y\Sigma mxy-\omega_z\Sigma mzx+\omega_x\Sigma mz^2,$$

or with the notation of Art. 255,

$$\Sigma m\,(y\dot{z}-z\dot{y})=A\omega_x-F\omega_y-E\omega_z.$$

Forming in the same way the angular momenta about the axes of y and z, we find the equations (20) in the form

$$\begin{aligned} A\omega_x-F\omega_y-E\omega_z&=\underset{\cdot}{H}_x,\\ -F\omega_x+B\omega_y-D\omega_z&=\underset{\cdot}{H}_y,\\ -E\omega_x-D\omega_y+C\omega_z&=\underset{\cdot}{H}_z. \end{aligned} \tag{2}$$

316. It appears, then, that *the rotation of angular velocity* ω *about the axis* l *can be regarded as due to an impulsive couple* $\underset{\cdot}{\mathrm{H}}$ *whose components are given by* (2). Conversely, the effect of a couple $\underset{\cdot}{H}$ on a rigid body at rest, with a fixed point, is to impart to the body a rotation ω whose magnitude and axis can be found by determining ω_x, ω_y, ω_z from (2).

Any system of impulses acting on the body can be reduced, for the fixed point O as origin, to a resultant $\underset{\cdot}{R}$ and a couple $\underset{\cdot}{H}$; the effect of the couple has just been stated; that of $\underset{\cdot}{R}$ consists merely in producing a pressure on the fixed point. To find this pressure, determine $\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ from (2); the velocity of the centroid can then be found and its components substituted in (1), Art. 314.

317. The axis l of the rotation produced by a given couple $\underset{\cdot}{H}$ is not, in general, perpendicular to the plane of the couple. Imagine the angular velocity ω to be represented by its rotor, *i.e.* by a length ω laid off from O on the axis l, and the couple $\underset{\cdot}{H}$ by its vector, *i.e.* by a length $\underset{\cdot}{H}$ laid off from O on the perpendicular to the plane of the couple. The relation between the rotor ω and the vector $\underset{\cdot}{H}$ producing it will best appear if we take the axis of rotation l as axis of z. We then have $\dot{x} = -\omega y$, $\dot{y} = \omega x$, $\dot{z} = 0$, and the momenta $-m\omega y$, $m\omega x$, 0 of the particles reduce to a resultant and couple at O as follows. The resultant momentum has the components:

$$-\omega\Sigma my = -M\bar{y}\omega, \quad \omega\Sigma mx = M\bar{x}\omega, \quad 0;$$

it is equal to $M\omega\sqrt{\bar{x}^2 + \bar{y}^2} = M\omega\bar{r}$, where $\bar{r}$ is the distance of the centroid from the axis l, and is perpendicular to the plane through axis and centroid. The couple has the components

$$-\omega\Sigma mzx = -E\omega, \quad -\omega\Sigma myz = -D\omega, \quad \omega\Sigma m(x^2 + y^2) = C\omega.$$

The equations (19) and (20) of Art. 238 reduce therefore to

$$-M\bar{y}\omega = \underset{\cdot}{R}_x + \underset{\cdot}{A}_x, \quad M\bar{x}\omega = \underset{\cdot}{R}_y + \underset{\cdot}{A}_y, \quad 0 = \underset{\cdot}{R}_z + \underset{\cdot}{A}_z, \tag{3}$$

$$-E\omega = \underset{\cdot}{H}_x, \quad -D\omega = \underset{\cdot}{H}_y, \quad C\omega = \underset{\cdot}{H}_z. \tag{4}$$

These equations can also be derived directly from the equations (1) and (2) above, since in the present case we have $\dot{\bar{x}} = -\omega\bar{y}$, $\dot{\bar{y}} = \omega\bar{x}$, $\dot{\bar{z}} = 0$, $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$.

318. The equations (4) show that, in general, the couple H has three components (see Fig. 37); H_x and H_y can be combined into a partial resultant $H_{xy} = \omega\sqrt{D^2+E^2}$ in the xy-plane; and the total resultant $H = \omega\sqrt{C^2+D^2+E^2}$ makes with the axis l an angle ϕ such that $\cos\phi = C/\sqrt{C^2+D^2+E^2}$. As C is always positive, this angle is always acute; it vanishes only if $D=0$ and $E=0$, *i.e.* if the instantaneous axis l is a principal axis at O.

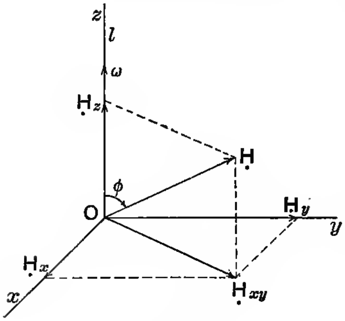

Fig. 37.

This result that H *and* ω *coincide only along a principal axis* is very important. It shows that *the vector* H *of the couple that produces a rotation* ω *has the direction of the axis of rotation* l *only, and always, if this axis* l *is a principal axis at the fixed point* O; *in this case we have* $H = I\omega$, *where* I *is the moment of inertia for* l.

Conversely, a couple H acting on a rigid body with a fixed point O produces rotation about an instantaneous axis l, which is, in general, inclined to the vector of the couple at an acute angle ϕ. This angle reduces to zero, *i.e.* the instantaneous axis l coincides in direction with the vector of the couple, only, and always, when the plane of the couple is perpendicular to a principal axis at O.

319. Let us now take the principal axes at O as axes of co-ordinates. Let ω_1, ω_2, ω_3 be the components of ω along these axes; H_1, H_2, H_3 those of H; and let I_1, I_2, I_3 be the principal moments, q_1, q_2, q_3 the principal radii of inertia at O. Then we must have

$$H_1 = I_1\omega_1 = Mq_1^2\omega_1, \quad H_2 = I_2\omega_2 = Mq_2^2\omega_2, \quad H_3 = I_3\omega_3 = Mq_3^2\omega_3. \quad (5)$$

These relations follow also from (2), since $A=I_1$, $B=I_2$, $C=I_3$, $D=0$, $E=0$, $F=0$; they determine the relation between H and ω in the general case.

320. The relation between the vectors $\underset{\cdot}{H}$ and ω is very clearly brought out by making use of the ellipsoids of inertia at the point O.

The reciprocal ellipsoid at O has the equation (see Arts. 269, 270)

$$\frac{x^2}{q_1^{\,2}}+\frac{y^2}{q_2^{\,2}}+\frac{z^2}{q_3^{\,2}}=1.$$

Let P (Fig. 38) be the point where it is met by the vector $\underset{\cdot}{H}$; x, y, z the co-ordinates, ρ the radius vector of P; hence x/ρ, y/ρ, z/ρ the direction-cosines of $\underset{\cdot}{H}$, so that $\underset{\cdot}{H}_1=\underset{\cdot}{H}x/\rho$, $\underset{\cdot}{H}_2=\underset{\cdot}{H}y/\rho$, $\underset{\cdot}{H}_3=\underset{\cdot}{H}z/\rho$. The equations (5) give, therefore,

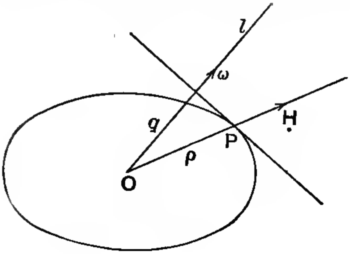

Fig. 38.

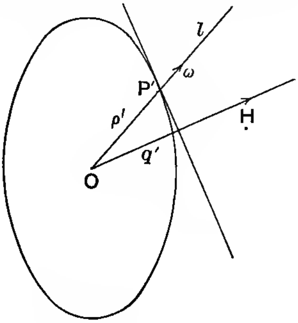

Fig. 39.

$$\omega_1=\frac{\underset{\cdot}{H}}{M\rho}\cdot\frac{x}{q_1^{\,2}},\quad \omega_2=\frac{\underset{\cdot}{H}}{M\rho}\cdot\frac{y}{q_2^{\,2}},\quad \omega_3=\frac{\underset{\cdot}{H}}{M\rho}\cdot\frac{z}{q_3^{\,2}},$$

whence

$$\omega=\frac{\underset{\cdot}{H}}{M\rho}\sqrt{\frac{x^2}{q_1^{\,4}}+\frac{y^2}{q_2^{\,4}}+\frac{z^2}{q_3^{\,4}}}=\frac{\underset{\cdot}{H}}{M\rho}\cdot\frac{1}{q}, \tag{6}$$

where q is the perpendicular let fall from O on the tangent plane at P (see Art. 269). The direction-cosines of ω,

$$\frac{\omega_1}{\omega}=q\frac{x}{q_1^{\,2}},\quad \frac{\omega_2}{\omega}=q\frac{y}{q_2^{\,2}},\quad \frac{\omega_3}{\omega}=q\frac{z}{q_3^{\,2}},$$

are the same as those of this perpendicular (*ib.*).

It thus appears that *the plane through* O *at right angles to the instantaneous axis* l *is conjugate to the direction of the vector* $\underset{\cdot}{\mathrm{H}}$ *with respect to the reciprocal ellipsoid at* O.

321. Again, the equation of the momental ellipsoid at O is (see Art. 266)

$$q_1^2x^2+q_2^2y^2+q_3^2z^2=\epsilon^4,$$

the semi-axes being $a=\epsilon^2/q_1$, $b=\epsilon^2/q_2$, $c=\epsilon^2/q_3$.

Let the instantaneous axis l meet this ellipsoid at a point P' (Fig. 39) whose co-ordinates and radius vector are x', y', z', ρ', so that x'/ρ', y'/ρ', z'/ρ' are the direction-cosines of l and $\omega_1=\omega x'/\rho'$, $\omega_2=\omega y'/\rho'$, $\omega_3=\omega z'/\rho'$. Substituting these values and introducing the semi-axes a, b, c, we find form (5)

$$\underset{\cdot}{H}_1=\frac{M\epsilon^4\omega}{\rho'}\cdot\frac{x'}{a^2},\qquad \underset{\cdot}{H}_2=\frac{M\epsilon^4\omega}{\rho'}\cdot\frac{y'}{b^2},\qquad \underset{\cdot}{H}_3=\frac{M\epsilon^4\omega}{\rho'}\cdot\frac{z'}{c^2},$$

whence $$\underset{\cdot}{H}=\frac{M\epsilon^4\omega}{\rho'}\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}=\frac{M\epsilon^4\omega}{\rho'}\cdot\frac{1}{q'},\tag{7}$$

where q' is the perpendicular let fall from O on the tangent plane at P'. The direction-cosines of $\underset{\cdot}{H}$,

$$\frac{\underset{\cdot}{H}_1}{\underset{\cdot}{H}}=q'\frac{x'}{a^2},\quad \frac{\underset{\cdot}{H}_2}{\underset{\cdot}{H}}=q'\frac{y'}{b^2},\quad \frac{\underset{\cdot}{H}_3}{\underset{\cdot}{H}}=q'\frac{z'}{c^2},$$

agree with those of q'.

It follows that *the plane of the couple* $\underset{\cdot}{\mathrm{H}}$ *is conjugate to the direction of the instantaneous axis* l *with respect to the momental ellipsoid at* O.

322. The **kinetic energy** of a rigid body with a fixed point O has the expression

$$T=\Sigma\tfrac{1}{2}\,mv^2=\tfrac{1}{2}\,\omega^2\Sigma mr^2,$$

where Σmr^2 is the moment of inertia for the instantaneous axis l.

Now, by Art. 270, the radius of inertia for the line l is equal to the distance of O from the perpendicular tangent plane to the reciprocal ellipsoid, *i.e.* to q (Fig. 38). Hence

$$T=\tfrac{1}{2}\,Mq^2\omega^2.\tag{8}$$

As, according to the fundamental property of the momental ellipsoid (see (16), Art. 266), we have $q=\epsilon^2/\rho'$, this becomes

$$T=\tfrac{1}{2}M\epsilon^4\cdot\left(\frac{\omega}{\rho'}\right)^2. \tag{9}$$

On the other hand, by Art. 258, if α, β, γ be the direction-cosines of l, *i.e.* of the rotor ω, we have

$$Mq^2=A\alpha^2+B\beta^2+C\gamma^2-2\,D\beta\gamma-2\,E\gamma\alpha-2\,F\alpha\beta\,;$$

hence

$$T=\tfrac{1}{2}(A\omega_z{}^2+B\omega_y{}^2+C\omega_z{}^2-2\,D\omega_y\omega_z-2\,E\omega_z\omega_z-2\,F\omega_z\omega_y). \tag{10}$$

Differentiating with respect to ω_x, ω_y, ω_z, and comparing with the equations (2), we find

$$\begin{aligned}\frac{\partial T}{\partial\omega_x}&=\quad A\omega_z-F\omega_y-E\omega_z=\underset{\cdot}{H}_x,\\ \frac{\partial T}{\partial\omega_y}&=-F\omega_z+B\omega_y-D\omega_z=\underset{\cdot}{H}_y,\\ \frac{\partial T}{\partial\omega_z}&=-E\omega_z-D\omega_y+C\omega_z=\underset{\cdot}{H}_z.\end{aligned} \tag{11}$$

If these relations be multiplied by ω_z, ω_y, ω_z and added, they give

$$2\,T=\frac{\partial T}{\partial\omega_x}\omega_z+\frac{\partial T}{\partial\omega_y}\omega_y+\frac{\partial T}{\partial\omega_z}\omega_z=\underset{\cdot}{H}_z\omega_z+\underset{\cdot}{H}_y\omega_y+\underset{\cdot}{H}_z\omega_z. \tag{12}$$

Substituting in the last expression $\alpha\omega=\omega_z$, $\beta\omega=\omega_y$, $\gamma\omega=\omega_z$, or $\lambda\underset{\cdot}{H}=\underset{\cdot}{H}_x$, $\mu\underset{\cdot}{H}=\underset{\cdot}{H}_y$, $\nu\underset{\cdot}{H}=\underset{\cdot}{H}_z$, where λ, μ, ν are the direction-cosines of the vector $\underset{\cdot}{H}$, we find

$$T=\tfrac{1}{2}\,\omega(\underset{\cdot}{H}_x\alpha+\underset{\cdot}{H}_y\beta+\underset{\cdot}{H}_z\gamma)=\tfrac{1}{2}\,\underset{\cdot}{H}(\omega_x\lambda+\omega_y\mu+\omega_z\nu). \tag{13}$$

We have, therefore, for the projection of $\underset{\cdot}{H}$ on the instantaneous axis l,

$$\underset{\cdot}{H}\cos\phi=\underset{\cdot}{H}_z\alpha+\underset{\cdot}{H}_y\beta+\underset{\cdot}{H}_z\gamma=\frac{2\,T}{\omega}=Mq^2\omega\,;$$

for the projection of ω on the direction of the vector $\underset{\cdot}{H}$,

$$\omega\cos\phi=\omega_x\lambda+\omega_y\mu+\omega_z\nu=\frac{2\,T}{\underset{\cdot}{H}}=Mq^2\frac{\omega^2}{\underset{\cdot}{H}}\,;$$

and finally,

$$T=\tfrac{1}{2}\,\underset{\cdot}{H}\omega\cos\phi. \tag{14}$$

323. If the principal axes at O be taken as axes of co-ordinates, we have to write $\omega_1, \omega_2, \omega_3$ for $\omega_x, \omega_y, \omega_z$; $\underset{\cdot}{H}_1, \underset{\cdot}{H}_2, \underset{\cdot}{H}_3$ for $\underset{\cdot}{H}_x, \underset{\cdot}{H}_y, \underset{\cdot}{H}_z$; I_1, I_2, I_3 for A, B, C, while $D=0$, $E=0$, $F=0$. Thus the relations (5) give $\underset{\cdot}{H}_1=I_1\omega_1$, $\underset{\cdot}{H}_2=I_2\omega_2$, $\underset{\cdot}{H}_3=I_3\omega_3$, whence

$$\underset{\cdot}{H}^2=I_1^2\omega_1^2+I_2^2\omega_2^2+I_3^2\omega_3^2, \tag{15}$$

$$\omega^2=\frac{\underset{\cdot}{H}_1^2}{I_1^2}+\frac{\underset{\cdot}{H}_2^2}{I_2^2}+\frac{\underset{\cdot}{H}_3^2}{I_3^2}; \tag{16}$$

and

$$T=\tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2) \tag{17}$$

$$=\tfrac{1}{2}\left(\frac{\underset{\cdot}{H}_1^2}{I_1}+\frac{\underset{\cdot}{H}_2^2}{I_2}+\frac{\underset{\cdot}{H}_3^2}{I_3}\right). \tag{18}$$

For the angle ϕ between $\underset{\cdot}{H}$ and ω, we have

$$\cos\phi=\frac{\underset{\cdot}{H}_1\omega_1+\underset{\cdot}{H}_2\omega_2+\underset{\cdot}{H}_3\omega_3}{\underset{\cdot}{H}\omega}=\frac{I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2}{\underset{\cdot}{H}\omega}. \tag{19}$$

2. CONTINUOUS MOTION UNDER ANY FORCES.

324. We now proceed to consider the motion of a rigid body with a fixed point when acted upon by any forces.

For the fixed point O as origin, the external forces reduce to a resultant R and a couple H. While the force R is taken up by the fixed point, the effect of the couple consists in changing the angular velocity ω about the instantaneous axis l, which exists at the time t, to the angular velocity $\omega+d\omega$ about another instantaneous axis l', which determines the motion of the body at the time $t+dt$. The point O being fixed, both axes, l and l', pass through it; and by Part I., Art. 303, the acceleration of any point (x, y, z) of the body has the following components parallel to rectangular axes fixed in the body and moving with it:

$$\begin{aligned}\ddot{x}&=\omega_x(\omega_x x+\omega_y y+\omega_z z)-\omega^2 x+\dot{\omega}_y z-\dot{\omega}_z y,\\ \ddot{y}&=\omega_y(\omega_x x+\omega_y y+\omega_z z)-\omega^2 y+\dot{\omega}_z x-\dot{\omega}_x z,\\ \ddot{z}&=\omega_z(\omega_x x+\omega_y y+\omega_z z)-\omega^2 z+\dot{\omega}_x y-\dot{\omega}^y x.\end{aligned} \tag{1}$$

Multiplying these expressions by the mass m of the particle situated at the point (x, y, z), we have the components of the effective force of this particle.

325. To form the equations of motion (4), Art. 223, and (6), Art. 224, for our case, we must reduce the system of the effective forces to its resultant and resulting couple; or, what amounts to the same thing, we must form the sums occurring in the left-hand members of these equations.

The summation of the components of the effective forces throughout the body gives, as usual,

$$\Sigma m\ddot{x} = M\ddot{\bar{x}},\quad \Sigma m\ddot{y} = M\ddot{\bar{y}},\quad \Sigma m\ddot{z} = M\ddot{\bar{z}},$$

where $\ddot{\bar{x}}$, $\ddot{\bar{y}}$, $\ddot{\bar{z}}$ are the components of the acceleration of the centroid. The resultant is therefore equal to the effective force of the centroid, the whole mass M of the body being regarded as concentrated at this point.

To make the body free, the reaction A of the fixed point should be introduced. Denoting its components by A_x, A_y, A_z, those of the resultant R of the external forces by R_x, R_y, R_z, the equations (4), Art. 223, assume the form

$$M\ddot{\bar{x}} = R_x + A_x,\quad M\ddot{\bar{y}} = R_y + A_y,\quad M\ddot{\bar{z}} = R_z + A_z. \tag{2}$$

The left-hand members evidently vanish if the origin be the centroid. The equations (2) can serve to determine the pressure $-A$ on the fixed point in magnitude and direction.

326. To form the moment $\Sigma m(y\ddot{z} - z\ddot{y})$ of the effective forces about the axis of x, we have to multiply the second of the expressions (1) by z, and subtract the product from the third multiplied by y; then multiply the difference by m, and sum throughout the body.

Performing this operation first on the last two terms which were shown in Part I., Art. 302, to be due to the angular acceleration, we find

$$\dot{\omega}_x \Sigma m(y^2 + z^2) - \dot{\omega}_y \Sigma mxy - \dot{\omega}_z \Sigma mzx = A\dot{\omega}_x - F\dot{\omega}_y - E\dot{\omega}_z,$$

with the notation of Art. 255. As the axes are fixed in the body (Art. 324), the moments and products of inertia are constant; and it appears from the equations (2), Art. 315, that this expression is the derivative with respect to the time of the component $\underset{.}{H}_x$ of the impulsive couple $\underset{.}{H}$ that produces the rotation ω at the time t.

Next operating in the same way on the remaining terms of the component accelerations (1), viz. those arising from the centripetal acceleration, we find

$$\begin{aligned}
&\omega_z(\omega_x\Sigma mxy+\omega_y\Sigma my^2+\omega_z\Sigma myz)-\omega^2\Sigma myz\\
&-\omega_y(\omega_x\Sigma mzx+\omega_y\Sigma myz+\omega_z\Sigma mz^2)+\omega^2\Sigma myz\\
&=\omega_z(F\omega_x+C\omega_y+D\omega_z)-\omega_y(E\omega_x+D\omega_y+B\omega_z)\\
&=\omega_y(-E\omega_x-D\omega_y+C\omega_z)-\omega_z(-F\omega_x+B\omega_y-D\omega_z)\\
&=\omega_y\underset{.}{H}_z-\omega_z\underset{.}{H}_y,
\end{aligned}$$

by (2), Art. 315.

The moments of the effective forces about the other two axes can now be obtained by cyclical permutation of the subscripts x, y, z. Thus we find that the equations (6), Art. 224, assume the form

$$\begin{aligned}
&\frac{d\underset{.}{H}_x}{dt}+\omega_y\underset{.}{H}_z-\omega_z\underset{.}{H}_y=H_x,\\
&\frac{d\underset{.}{H}_y}{dt}+\omega_z\underset{.}{H}_x-\omega_x\underset{.}{H}_z=H_y, \qquad (3)\\
&\frac{d\underset{.}{H}_z}{dt}+\omega_x\underset{.}{H}_y-\omega_y\underset{.}{H}_x=H_z.
\end{aligned}$$

The reaction A of the fixed point does not enter into these equations; as it intersects every one of the axes, its moments about these axes are zero.

327. Geometrically the equations (3) mean that the vector H of the resultant couple of the external forces has two components one of which resolves itself along the axes into $d\underset{.}{H}_x/dt$, $d\underset{.}{H}_y/dt$, $d\underset{.}{H}_z/dt$, while the other has the components $\omega_y \underset{\cdot}{H}_z - \omega_z \underset{\cdot}{H}_y$, $\omega_z \underset{\cdot}{H}_x - \omega_x \underset{\cdot}{H}_z$, $\omega_x \underset{\cdot}{H}_y - \omega_y \underset{\cdot}{H}_x$. Each of these components can be interpreted geometrically, if we imagine the vector $\underset{\cdot}{H}$ of the impulsive couple drawn from O as origin, so that the co-ordinates of its extremity are $\underset{\cdot}{H}_x$, $\underset{\cdot}{H}_y$, $\underset{\cdot}{H}_z$ (Fig. 40). The time-derivatives of these co-ordinates are the velocities of the extremity of the vector $\underset{\cdot}{H}$ with respect to the axes of co-ordinates which, it will be remembered, are fixed in the body. Hence that component of H which is due to the angular acceleration is the relative velocity of the extremity of $\underset{\cdot}{H}$ with respect to the body.

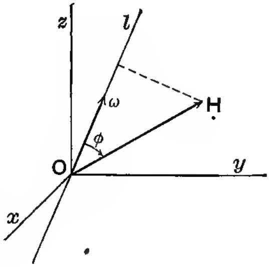

Fig. 40.

The other component, which is due to the centripetal acceleration, evidently represents the linear velocity, arising from the angular velocity ω, of the point of the body that coincides at the time t with the same extremity of the vector $\underset{\cdot}{H}$.

It follows that the vector H represents in magnitude and direction the absolute velocity of the extremity of the vector $\underset{\cdot}{H}$; in other words, H is geometrically equal to the geometrical increment of $\underset{\cdot}{H}$ divided by the element of time. This was to be expected, and might indeed be taken as starting-point for deriving the equations (3).

328. Let us now select as axes of co-ordinates the principal axes at O. According to our usual notation, we have then to exchange the subscripts x, y, z for 1, 2, 3. Moreover, as shown in Art. 319, $\underset{\cdot}{H}_1 = I_1\omega_1$, $\underset{\cdot}{H}_2 = I_2\omega_2$, $\underset{\cdot}{H}_3 = I_3\omega_3$, where I_1, I_2, I_3 are the principal moments of inertia at O. Thus the equations (3) reduce to the following:

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= H_1, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= H_2, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= H_3. \end{aligned} \tag{4}$$

These are **Euler's equations of motion.** Their solution gives ω_1, ω_2, ω_3 as functions of the time t.

It may be noted that the equations (4) are often written with the following notation:

$$
\begin{aligned}
A\frac{dp}{dt}+(C-B)qr&=L,\\
B\frac{dq}{dt}+(A-C)rp&=M, \qquad (4')\\
C\frac{dr}{dt}+(B-A)pq&=N,
\end{aligned}
$$

where A, B, C are the principal moments of inertia; p, q, r the components of the angular velocity ω along the principal axes; L, M, N the components of the resulting couple H of the external forces along the same axes.

329. Owing to the importance of the equations (4) it may be well to indicate another way of deriving them.

The rotation of angular velocity ω about the instantaneous axis l during the first element of time can be regarded as due to an impulsive couple H (Art. 316). Even if there were no external forces acting, the body would not in general continue to turn with the same velocity about the same axis. For if this were the case, any particle m of the body, at the distance r from the axis l would be moving uniformly in a circle of radius r, with a velocity ωr, and such uniform circular motion requires for its maintenance the action of a centripetal force.

Let us therefore introduce at every particle m two equal and opposite forces (Fig. 41), the centripetal force $m\omega^2 r$ directed towards the axis l, and the centrifugal force $-m\omega^2 r$; the introduction of these forces does not change the state of motion of the body.

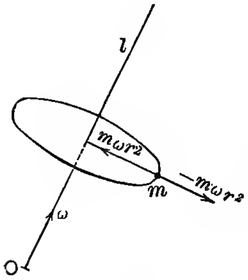

Fig. 41.

330. If the system of centripetal forces $m\omega^2 r$ alone were introduced and no other forces were acting, the body would continue to turn with the same angular velocity ω about the same axis l. The effect of the system of centrifugal forces $-m\omega^2 r$ represents therefore the change that would take place in the motion if no external forces were acting.

Let us reduce these centrifugal forces to their resultant and resulting couple, the fixed point O being taken as origin and the axis l as axis of z. In doing this we can make use of the reduction of momenta in Art. 317. For, evidently, the vector representing the centrifugal force $-m\omega^2 r$ can be obtained by multiplying the momentum $m\omega r$ of the particle m by ω and turning it through an angle of 90° in a sense opposite to that of the rotation ω. The reduction to O gives therefore a resultant force $M\omega^2\bar{r}$, in the xy-plane, directed toward the projection of the centroid on this plane. The resulting couple has its z-component equal to zero since all the centrifugal forces intersect the axis of z; the vector of the resulting couple lies therefore in the xy-plane, has the magnitude $\omega \underset{\cdot}{H}_{xy}$, and is perpendicular to the $\underset{\cdot}{H}_{xy}$ in Fig. 37.

331. The resultant vanishes only if $\bar{r} = 0$, *i.e.* if the centroid lies on the axis l; the couple vanishes if $\omega \underset{\cdot}{H}_{xy} \equiv \omega^2\sqrt{D^2 + E^2} = 0$, *i.e.* if the axis l is a principal axis at O. It follows that the centrifugal forces reduce to zero only if the axis of rotation is a principal centroidal axis; in this case the direction of the axis remains unchanged.

By Art. 318 (see Fig. 37) we have $\underset{\cdot}{H}_{xy} = \underset{\cdot}{H} \sin\phi$; hence the resulting couple of the centrifugal forces $= \omega \underset{\cdot}{H} \sin\phi$, that is, its magnitude is represented by the area of the parallelogram formed by the vectors $\underset{\cdot}{H}$ and ω; the vector of this couple as shown above is perpendicular to this area. Projecting this parallelogram on any three rectangular co-ordinate planes, with O as origin, we find, since ω_x, ω_y, ω_z are the co-ordinates of the extremity of the rotor ω, $\underset{\cdot}{H}_x$, $\underset{\cdot}{H}_y$, $\underset{\cdot}{H}_y$ those of the extremity of the vector $\underset{\cdot}{H}$ drawn from O:

$$\omega_z \underset{\cdot}{H}_y - \omega_y \underset{\cdot}{H}_z, \quad \omega_x \underset{\cdot}{H}_z - \omega_z \underset{\cdot}{H}_x, \quad \omega_y \underset{\cdot}{H}_x - \omega_x \underset{\cdot}{H}_y.$$

This agrees with the results found in Arts. 326, 327.

332. If the principal axes at O be taken as axes of co-ordinates, the components of the resultant couple of the centrifugal forces become

$$\omega_3 \underset{\cdot}{H}_2 - \omega_2 \underset{\cdot}{H}_3, \quad \omega_1 \underset{\cdot}{H}_3 - \omega_3 \underset{\cdot}{H}_1, \quad \omega_2 \underset{\cdot}{H}_1 - \omega_1 \underset{\cdot}{H}_2,$$

or, since $\underset{\cdot}{H}_1 = I_1\omega_1$, $\underset{\cdot}{H}_2 = I_2\omega_2$, $\underset{\cdot}{H}_3 = I_3\omega_3$,

$$(I_2 - I_3)\,\omega_2\omega_3, \quad (I_3 - I_1)\,\omega_3\omega_1, \quad (I_1 - I_2)\,\omega_1\omega_2.$$

As the planes of these couples are perpendicular to the principal axes at O, they produce during the element of time infinitesimal rotations about these axes, whose angles are, by Art. 318:

$$\frac{I_2 - I_3}{I_1}\,\omega_2\omega_3\,dt, \quad \frac{I_3 - I_1}{I_2}\,\omega_3\omega_1\,dt, \quad \frac{I_1 - I_2}{I_3}\,\omega_1\omega_2\,dt.$$

These are the only increments of ω_1, ω_2, ω_3 if there are no forces acting on the body; hence, in this case we must have

$$I_1\frac{d\omega_1}{dt}=(I_2-I_3)\,\omega_2\omega_3,\quad I_2\frac{d\omega_2}{dt}=(I_3-I_1)\,\omega_3\omega_1,\quad I_3\frac{d\omega_3}{dt}=(I_1-I_2)\,\omega_1\omega_2.$$

If, however, there are external forces acting on the body, whose resulting couple for O is H, with the components H_1, H_2, H_3 along the principal axes at O, these couples produce infinitesimal rotations

$$\frac{H_1}{I_1}dt,\quad \frac{H_2}{I_2}dt,\quad \frac{H_3}{I_3}dt,$$

and the equations of motion are therefore

$$\begin{aligned}I_1\dot\omega_1&=(I_2-I_3)\,\omega_2\omega_3+H_1,\\ I_2\dot\omega_2&=(I_3-I_1)\,\omega_3\omega_1+H_2,\\ I_3\dot\omega_3&=(I_1-I_2)\,\omega_1\omega_2+H_3.\end{aligned}$$

These are Euler's equations (4).

333. Euler's equations determine the angular velocities of the body about the principal axes which move with the body. The position of these moving axes with respect to a system of fixed rectangular axes through the fixed point O can be expressed by means of three angles.

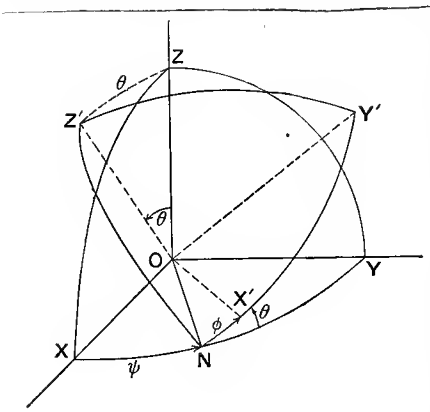

Fig. 42.

Let X, Y, Z (Fig. 42) be the intersections of the fixed axes, with a sphere of radius one, described about O as centre; X', Y', Z' those of the moving principal axes; N the intersection with the same sphere of the so-called *line of nodes*, *i.e.* the line in which the planes XOY and $X'OY'$ intersect. Then the angles

$$ZZ'=\theta,\quad NX'=\phi,\quad XN=\psi,$$

usually called **Euler's angles**, may serve to determine the relation between the two systems of axes.

334. The *sense* in which these angles are counted is best remembered by imagining the two trihedral angles XYZ and $X'Y'Z'$ originally coincident. Now turn the system $X'Y'Z'$ about the axis OZ in the positive sense (counter-clockwise) until the axis OX' coincides with the assumed positive sense of the nodal line ON, *i.e.* the final intersection of the planes XOY and $X'OY'$; the amount of this rotation gives the angle ψ. Next turn the trihedral $X'Y'Z'$ about this line of nodes in the positive sense until the plane $X'OY'$ falls into its final position; this gives the angle θ, as the angle between the planes XOY and $X'OY'$ at N, or as the angle ZOZ' between their normals. Finally a rotation of $X'Y'Z'$ about the axis OZ', which has reached its final position, in the positive sense until OX' comes into its final position, determines the angle ϕ.

335. The angular velocity, represented by its rotor ω, whose components along OX', OY', OZ' are ω_1, ω_2, ω_3, can be resolved along ON, OZ', OZ into three components which are evidently $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, respectively. The sum of the projections of these three components on the line OX' should give ω_1; hence

$$\omega_1 = \dot{\theta}\cos\phi + \dot{\phi}\cos\tfrac{1}{2}\pi + \dot{\psi}\cos ZX'.$$

Similarly
$$\omega_2 = \dot{\theta}\cos(\phi + \tfrac{1}{2}\pi) + \dot{\phi}\cos\tfrac{1}{2}\pi + \dot{\psi}\cos ZY',$$
$$\omega_3 = \dot{\theta}\cos\tfrac{1}{2}\pi + \dot{\phi}\cos 0 + \dot{\psi}\cos\theta.$$

The spherical triangle ZNX' gives (by the fundamental formula of spherical trigonometry, $\cos c = \cos a\cos b + \sin a\sin b\cos\gamma$) $\cos ZX' = \sin\phi\cos(\tfrac{1}{2}\pi - \theta) = \sin\phi\sin\theta$; and the triangle ZNY' gives $\cos ZY' = \sin(\phi + \tfrac{1}{2}\pi)\cos(\tfrac{1}{2}\pi - \theta) = \cos\phi\sin\theta$. Hence, finally

$$\begin{aligned}\omega_1 &= \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\sin\theta,\\ \omega_2 &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\sin\theta,\\ \omega_3 &= \dot{\phi} + \dot{\psi}\cos\theta.\end{aligned} \tag{5}$$

Solving these equations for $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, we find

$$\begin{aligned}\dot{\theta} &= \omega_1\cos\phi - \omega_2\sin\phi,\\ \dot{\phi} &= -\omega_1\sin\phi\cot\theta - \omega_2\cos\phi\cot\theta + \omega_3,\\ \dot{\psi} &= \omega_1\sin\phi\csc\theta + \omega_2\cos\phi\csc\theta.\end{aligned} \tag{6}$$

336. The relation between two rectangular systems of axes with the same origin can also be expressed by means of the 9 cosines of the angles between the axes.

Let O be the common origin, x, y, z the co-ordinates of any point with respect to the fixed system, x', y', z' its co-ordinates in the moving system; then we have, evidently,

$$\begin{aligned} x &= a_1x' + a_2y' + a_3z', \\ y &= b_1x' + b_2y' + b_3z', \\ z &= c_1x' + c_2y' + c_3z', \end{aligned} \qquad (7)$$

where the coefficients of x', y', z' are the cosines of the angles between the axes, which can best be remembered in the form

	x'	y'	z'
x	a_1	a_2	a_3
y	b_1	b_2	b_3
z	c_1	c_2	c_3

(8)

Thus, 9 angles are used in order to fix the position of the moving axes with respect to the fixed axes, instead of Euler's 3 angles. But their 9 cosines (8) are connected by 6 independent relations, which can be written in either one of the equivalent forms:

$$\begin{aligned} a_1^2 + b_1^2 + c_1^2 &= 1, & a_2a_3 + b_2b_3 + c_2c_3 &= 0, \\ a_2^2 + b_2^2 + c_2^2 &= 1, & a_3a_1 + b_3b_1 + c_3c_1 &= 0, \\ a_3^2 + b_3^2 + c_3^2 &= 1, & a_1a_2 + b_1b_2 + c_1c_2 &= 0, \end{aligned} \qquad (9)$$

or

$$\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= 1, & b_1c_1 + b_2c_2 + b_3c_3 &= 0, \\ b_1^2 + b_2^2 + b_3^2 &= 1, & c_1a_1 + c_2a_2 + c_3a_3 &= 0, \\ c_1^2 + c_2^2 + c_3^2 &= 1, & a_1b_1 + a_2b_2 + a_3b_3 &= 0. \end{aligned} \qquad (10)$$

The meaning of these equations is easily perceived from the meaning of the angles involved. Thus, the first of the equations, (9) expresses the fact that a_1, b_1, c_1 are the direction-cosines of a line, viz. the axis Ox'; the last of the equations (10) expresses the perpendicularity of the axes Ox and Oy; and similarly for the others.

337. The relations between the 9 angles, whose cosines are given in (8), and Euler's 3 angles θ, ϕ, ψ are readily found from Fig. 42, by applying the formula $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$ successively to the triangles

$$\begin{array}{lll} XNX', & XNY', & XNZ', \\ YNX', & YNY', & YNZ', \\ ZNX', & ZNY', & ZNZ'. \end{array}$$

In this way the following relations are found:

$$a_1 = \cos\psi \cos\phi - \sin\psi \sin\phi \cos\theta,$$
$$b_1 = \sin\psi \cos\phi + \cos\psi \sin\phi \cos\theta,$$
$$c_1 = \sin\phi \sin\theta,$$

$$\begin{array}{ll} a_2 = -\cos\psi \sin\phi - \sin\psi \cos\phi \cos\theta, & a_3 = \sin\psi \sin\theta, \\ b_2 = -\sin\psi \sin\phi + \cos\psi \cos\phi \cos\theta, & b_3 = -\cos\psi \sin\theta, \\ c_2 = \cos\phi \sin\theta, & c_3 = \cos\theta. \end{array}$$

338. It is evident, geometrically, from (8), that we must have

$$\begin{aligned} x' &= a_1 x + b_1 y + c_1 z, \\ y' &= a_2 x + b_2 y + c_2 z, \\ z' &= a_3 x + b_3 y + c_3 z. \end{aligned} \tag{11}$$

For just as the first of the equations (7) expresses that the sum of the projections on Ox of the co-ordinates x', y', z' is equal to x, so the first of the equations (11) expresses the equality of x' to the sum of the projections of x, y, z on the axis Ox'; and similarly for the other equations.

Now the solution of the equations (7) for x', y', z' should give the values (11). Putting

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Delta,$$

solving the equations (7) for x', and comparing the coefficients of x, y, z to those in (11), we find the following relations:

$$a_1 = \frac{b_2c_3 - b_3c_2}{\Delta}, \quad b_1 = \frac{c_2a_3 - c_3a_2}{\Delta}, \quad c_1 = \frac{a_2b_3 - a_3b_2}{\Delta}. \qquad (12)$$

339. Squaring and adding these equations and applying the relations (9), we find after reduction

$$\Delta^2 = 1.$$

The two values of Δ, $+1$ and -1, correspond to the two different relations between the two rectangular systems, which might perhaps be called like and unlike. Two systems are alike if their positive axes can be brought to coincidence; they are unlike if this cannot be done. It is, of course, always possible to bring the axes Ox' and Oy' to coincidence with Ox and Oy, respectively. But after having accomplished this, the axis Oz' may fall along Oz, in which case the systems are alike, or it may fall into the opposite direction, when the systems are unlike.

Now if Ox' coincides with Ox, Oy' with Oy, we have $a_1 = 1$, $b_2 = 1$; and $c_3 = +1$ for like systems, $c_3 = -1$ for unlike systems; as the other 6 cosines are zero, we find that $\Delta = +1$ corresponds to like systems, and $\Delta = -1$ to unlike systems. For it is evident that the motion of one system with respect to the other cannot affect Δ so as to change from one of these values to the other.

In mechanics the systems should generally be such that they can be brought to coincidence. We assume therefore $\Delta = 1$.

With this value of Δ, the equations (12) and the similar relations obtained by cyclical permutation of the subscripts give the identities:

$$\begin{array}{lll} a_1 = b_2c_3 - b_3c_2, & b_1 = c_2a_3 - c_3a_2, & c_1 = a_2b_3 - a_3b_2, \\ a_2 = b_3c_1 - b_1c_3, & b_2 = c_3a_1 - c_1a_3, & c_2 = a_3b_1 - a_1b_3, \\ a_3 = b_1c_2 - b_2c_1, & b_3 = c_1a_2 - c_2a_1, & c_3 = a_1b_2 - a_2b_1. \end{array} \qquad (13)$$

340. If the axes Ox', Oy', Oz' be the principal axes at O, the equations (7) exhibit the relations between the system of the principal axes and a fixed system with the same origin O, by means of the cosines of the 9 angles between the axes of the two systems. They can be used to derive Euler's equations by a purely analytical process from the equations (7), Art. 224.

To accomplish this we must form the quantities $\Sigma m(y\dot{z}-z\dot{y})$, $\Sigma m(z\dot{x}-x\dot{z})$, $\Sigma m(x\dot{y}-y\dot{x})$. We need therefore $\dot{x}$, $\dot{y}$, $\dot{z}$. Now, differentiating the expressions (7) with respect to the time and remembering that x', y', z' are independent of the time, we find:

$$\dot{x}=\dot{a}_1x'+\dot{a}_2y'+\dot{a}_3z', \quad \dot{y}=\dot{b}_1x'+\dot{b}_2y'+\dot{b}_3z',$$
$$\dot{z}=\dot{c}_1x'+\dot{c}_2y'+\dot{c}_3z'. \tag{14}$$

To introduce the angular velocities ω_1, ω_2, ω_3 about the principal axes at O, we observe that the direction-cosines a_1, a_2, a_3 of the axis Ox can be regarded as the co-ordinates of the point situated on Ox at unit distance from O. The components of the linear velocity of this point, arising from ω_1, ω_2, ω_3 are

$$\dot{a}_1=a_2\omega_3-a_3\omega_2, \quad \dot{a}_2=a_3\omega_1-a_1\omega_3, \quad \dot{a}_3=a_1\omega_2-a_2\omega_1;$$

and similarly we have for points at unit distance from O on Oy and Oz:

$$\dot{b}_1=b_2\omega_3-b_3\omega_2, \quad \dot{b}_2=b_3\omega_1-b_1\omega_3, \quad \dot{b}_3=b_1\omega_2-b_2\omega_1,$$
$$\dot{c}_1=c_2\omega_3-c_3\omega_2, \quad \dot{c}_2=c_3\omega_1-c_1\omega_3, \quad \dot{c}_3=c_1\omega_2-c_2\omega_1.$$

It should be noticed that the motion of a body with a fixed point is fully determined by the motion of two of its points, not in the same line with the fixed point; the third point is here only introduced to preserve the symmetry.

Substituting these values in (14), we find

$$\begin{aligned}\dot{x}&=(a_2\omega_3-a_3\omega_2)x'+(a_3\omega_1-a_1\omega_3)y'+(a_1\omega_2-a_2\omega_1)z',\\ \dot{y}&=(b_2\omega_3-b_3\omega_2)x'+(b_3\omega_1-b_1\omega_3)y'+(b_1\omega_2-b_2\omega_1)z',\\ \dot{z}&=(c_2\omega_3-c_3\omega_2)x'+(c_3\omega_1-c_1\omega_3)y'+(c_1\omega_2-c_2\omega_1)z'.\end{aligned} \tag{15}$$

341. From (7) and (15) we now find, if we remember that $\Sigma my'z'=0$, $\Sigma mz'x'=0$, $\Sigma mx'y'=0$, since Ox', Oy', Oz' are principal axes at O:

$$\Sigma m(y\dot{z}-z\dot{y})=(b_2c_3-b_3c_2)\omega_1\Sigma m(y'^2+z'^2)$$
$$+(b_3c_1-b_1c_3)\omega_2\Sigma m(z'^2+x'^2)+(b_1c_2-b_2c_1)\omega_3\Sigma m(x'^2+y'^2),$$

or applying the relations (13) and denoting the principal moments of inertia at O by I_1, I_2, I_3:

$$\Sigma m(y\dot{z}-z\dot{y})=a_1I_1\omega_1+a_2I_2\omega_2+a_3I_3\omega_3.$$

The quantities $\Sigma m(z\dot{x}-x\dot{z})$ and $\Sigma m(x\dot{y}-y\dot{x})$ are obtained from this result by cyclical permutation of the letters a, b, c.

Thus the equations (7), Art. 224, assume the form:

$$\begin{aligned}
&\frac{d}{dt}(a_1I_1\omega_1+a_2I_2\omega_2+a_3I_3\omega_3)=H_x,\\
&\frac{d}{dt}(b_1I_1\omega_1+b_2I_2\omega_2+b_3I_3\omega_3)=H_y, \qquad (16)\\
&\frac{d}{dt}(c_1I_1\omega_1+c_2I_2\omega_2+c_3I_3\omega_3)=H_z.
\end{aligned}$$

The geometrical meaning of these equations is apparent. $I_1\omega_1$, $I_2\omega_2$, $I_3\omega_3$ are the components along the principal axes of the vector of the resultant impulsive couple $\underset{.}{H}$ (Art. 319); hence $a_1I_1\omega_1+a_2I_2\omega_2+a_3I_3\omega_3$ is the component of $\underset{.}{H}$ along the fixed axis Ox; the equations (16) express therefore the fact that H is geometrically equal to the geometrical derivative of $\underset{.}{H}$ with respect to the time (see Art. 327); they can be written in the form

$$\frac{d\underset{.}{H}_x}{dt}=H_x,\quad \frac{d\underset{.}{H}_y}{dt}=H_y,\quad \frac{d\underset{.}{H}_z}{dt}H_z.$$

342. If the equations (16) be multiplied first by a_1, b_1, c_1, then by a_2, b_2, c_2, finally by a_3, b_3, c_3, and each time added, the right-hand members of the resulting equations will evidently represent H_1, H_2, H_3, respectively, *i.e.* the components of H

along the principal axes at O. The left-hand members reduce also to a simple form if the differentiations indicated in (16) be performed, the values for $\dot{a}_1$, $\dot{a}_2$, $\dot{a}_3$, $\dot{b}_1$, $\cdots$ be substituted from Art. 340, and the relations (9) be applied. As final result we find Euler's equations:

$$\begin{aligned}
I_1\dot{\omega}_1+(I_3-I_2)\omega_2\omega_3&=H_1,\\
I_2\dot{\omega}_2+(I_1-I_3)\omega_3\omega_1&=H_2,\\
I_3\dot{\omega}_3+(I_2-I_1)\omega_1\omega_2&=H_3.
\end{aligned}$$

3. CONTINUOUS MOTION WITHOUT FORCES.

343. In the particular case when no external forces are acting, the motion of a rigid body about a fixed point admits of an elegant geometrical interpretation which is due to Poinsot.

As there are no external forces, we have $H=0$, and hence $\underset{\cdot}{H}$ is constant in magnitude and direction. The plane of this couple is the *invariable plane* (see Arts. 230–232) which always exists in the case of no forces; its vector indicates the *invariable direction.*

The body can be replaced by its momental ellipsoid at O, and the invariable plane can be imagined placed so as to be tangent to this ellipsoid at a point P' (Fig. 43). The radius vector $OP'=\rho'$ of the point of contact P' is the diameter of the ellipsoid conjugate to the invariable plane; hence the line OP' is the instantaneous axis l of the rotation (Art. 321).

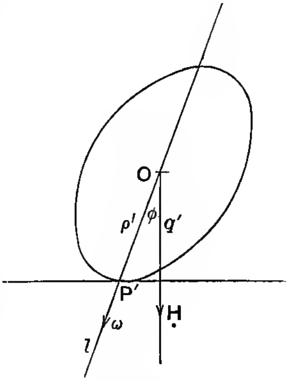

Fig. 43.

Now it can be shown that the perpendicular distance q' of O from the invariable plane (this plane being always placed so as to be tangent to the varying positions of the momental ellipsoid) is constant; it then follows at once that *the motion of the body consists in the rolling of its momental ellipsoid over the invariable plane.*

344. To prove that q' is constant, it should be remembered that, by (7), Art. 321, we have

$$q' = M\epsilon^4 \cdot \frac{\omega}{\rho'} \cdot \frac{1}{H}.$$

As H is constant in our case, it only remains to show that ω/ρ' is constant. This follows from the expression (9) for the kinetic energy T, given in Art. 322, viz.

$$T = \tfrac{1}{2} M\epsilon^4 \left(\frac{\omega}{\rho'}\right)^2;$$

for, as there are no external forces, no work is done, and the kinetic energy must remain constant; hence ω/ρ' is constant, and ω is directly proportional to ρ'.

Moreover, the expression (14) of Art. 322 shows that

$$\omega \cos \phi = \frac{2T}{H} = \text{const.}, \tag{1}$$

that is, the projection $\omega \cos \phi$ of the angular velocity ω on the invariable direction remains the same throughout the motion.

345. It has been pointed out in Part I., Art. 35, that the motion of a rigid body with a fixed point can always be regarded as produced by the rolling of the cone of the body axes over the cone of the space axes, these cones having their common vertex at the fixed point O. The body axes, *i.e.* the lines l' of the body that become instantaneous axes of rotation in the course of the motion, form a cone, invariably connected with the momental ellipsoid at O, and intersecting this ellipsoid in a curve fixed in the body. This curve has been called by Poinsot the *polhode* (or path of the instantaneous pole P', Fig. 43).

The cone of the space axes l, which is fixed in space, intersects the invariable plane in a curve called *herpolhode* (or creeping path of the pole). During the motion of the body, the polhode rolls over the herpolhode.

346. The equations of the polhode are easily obtained by considering that this curve is the locus of those points of the ellipsoid whose tangent plane has the constant distance q' from the centre O. Hence, denoting the semi-axes of the momental ellipsoid by a, b, c, the *equations of the polhode* are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{1}{q'^2}. \tag{2}$$

It can, therefore, be regarded as the intersection of the momental ellipsoid with a coaxial ellipsoid whose semi-axes are a^2/q', b^2/q', c^2/q'.

Multiplying the second equation by q'^2, and subtracting the result from the first equation, we find the equation of the *cone of the body axes*

$$\left(1-\frac{q'^2}{a^2}\right)\frac{x^2}{a^2}+\left(1-\frac{q'^2}{b^2}\right)\frac{y^2}{b^2}+\left(1-\frac{q'^2}{c^2}\right)\frac{z^2}{c^2}=0. \tag{3}$$

This is a cone of the second order, concentric and coaxial with the momental ellipsoid.

The polhode evidently consists of two equal separate branches, of which it is sufficient to consider one. Each branch has four vertices situated in the principal planes of the ellipsoid.

The herpolhode is confined between two concentric circles whose centre is the projection of O on the invariable plane. It is a transcendental curve and is in general not closed.

347. If the momental ellipsoid is an ellipsoid of revolution, the polhode consists of two circles, and the herpolhode is also a circle; as ρ' is in this case constant, it follows that ω remains constant.

If we assume in the general case $a > b > c$, the polhode reduces to two points whenever $q' = a$ or $q' = c$. The rotation then takes place about a principal axis and is *permanent*. If $q' = b$ (which does not necessarily mean that the axis of rotation coincides with the middle axis b), the cone of body axes reduces to two planes

$$\left(1-\frac{b^2}{a^2}\right)\frac{x^2}{a^2}+\left(1-\frac{b^2}{c^2}\right)\frac{z^2}{c^2}=0,$$

each of which intersects the ellipsoid in an ellipse. These ellipses divide the surface of the ellipsoid into two pairs of opposite regions, one about the greatest axis a, the other about the least c.

As long as $a > q' > b$, the polhode lies in the former region, and the cone of body axes has a as its axis. If $b > q' > c$, the polhode lies in the other region, and c is the axis of the cone.

Two polhodes cannot intersect; for if they did, the tangent plane at the point of intersection would have two different distances from the centre, which is impossible.

348. The motion of a body is called *stable* if after a slight disturbance the body tends to resume the original motion. In our case a slight disturbance displaces the instantaneous axis from one polhode to another near by. Hence if the polhode be situated very near to one of the bounding ellipses, the motion is not stable, because a slight disturbance might change the polhode to one in the other region. The motion is therefore the more stable the more closely the polhode surrounds either the greatest or the least axis of the ellipsoid.

If, however, one of the regions between the ellipses be very narrow, which will be the case if two of the axes of the ellipsoid are nearly equal, a polhode in this region, though close to the vertex, may still approach very near to the ellipses so as to make the motion unstable.

349. Integration of Euler's Equations. As $H=0$, Euler's equations (4), Art. 328, are

$$I_1\frac{d\omega_1}{dt}=(I_2-I_3)\,\omega_2\omega_3,\quad I_2\frac{d\omega_2}{dt}=(I_3-I_1)\,\omega_3\omega_1,$$

$$I_3\frac{d\omega_3}{dt}=(I_1-I_2)\omega_1\omega_2. \tag{4}$$

Multiplying by ω_1, ω_2, ω_3, and adding, we find

$$\frac{d}{dt}\cdot\tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)=0,$$

whence, by (17), Art. 323,

$$\tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)\equiv T=\text{const.} \tag{5}$$

This is nothing but the *equation of kinetic energy.*

Again, multiplying the equations (4) by $I_1\omega_1$, $I_2\omega_2$, $I_3\omega_3$, and adding, we find similarly, by (15), Art. 323,

$$I_1^2\omega_1^2+I_2^2\omega_2^2+I_3^2\omega_3^2\equiv H^2=\text{const.} \tag{6}$$

This is the *principle of areas* or *of the invariable plane.*

As, moreover,

$$\omega_1^2+\omega_2^2+\omega_3^2=\omega^2, \tag{7}$$

we have three equations (5), (6), (7) for determining ω_1^2, ω_2^2, ω_3^2. Their solution gives, after some reductions,

$$\omega_1^2=\frac{(I_2+I_3)\cdot 2T-H^2-I_2I_3\omega^2}{(I_1-I_2)(I_3-I_1)},\quad \omega_2^2=\frac{(I_3+I_1)\cdot 2T-H^2-I_3I_1\omega^2}{(I_2-I_3)(I_1-I_2)},$$

$$\omega_3^2=\frac{(I_1+I_2)\cdot 2T-H^2-I_1I_2\omega^2}{(I_3-I_1)(I_2-I_3)}. \tag{8}$$

350. To find the *time*, multiply the equations (4) by ω_1/I_1, ω_2/I_2, ω_3/I_3, and add. This gives

$$\frac{d}{dt}\tfrac{1}{2}(\omega_1^2+\omega_2^2+\omega_3^2)=\omega_1\omega_2\omega_3\left(\frac{I_2-I_3}{I_1}+\frac{I_3-I_1}{I_2}+\frac{I_1-I_2}{I_3}\right),$$

or

$$\frac{d}{dt}\tfrac{1}{2}\omega^2=-\frac{(I_2-I_3)(I_3-I_1)(I_1-I_2)}{I_1I_2I_3}\omega_1\omega_2\omega_3.$$

In this equation the values (8) should be substituted for ω_1, ω_2, ω_3. For the sake of brevity, let us put

$$(I_2+I_3)\cdot 2T-H^2=I_2I_3\alpha^2,\quad (I_3+I_1)\cdot 2T-H^2=I_3I_1\beta^2,$$

$$(I_1+I_2)\cdot 2T-H^2=I_1I_2\gamma^2;$$

we then find

$$\int dt=\pm\tfrac{1}{2}\int\frac{d(\omega^2)}{\sqrt{(\omega^2-\alpha^2)(\beta^2-\omega^2)(\omega^2-\gamma^2)}} \tag{9}$$

This is an elliptical integral whose discussion is beyond the scope of the present treatise.

351. It remains to determine the position of the moving system formed by the principal axes, with respect to a fixed system of axes through O, by means of Euler's angles θ, ϕ, ψ (Art. 333). After finding ω as a function of t from (9), we have, by (8), ω_1, ω_2, ω_3 as functions of t. Substituting these values into the equations (5) or (6), Art. 335, we have a system of differential equations of the first order whose integration gives θ, ϕ, ψ as functions of t.

352. To illustrate the method by a simple example, let us consider the case of *a body whose momental ellipsoid is an ellipsoid of revolution.*

Let $I_1 = I_2$; then, putting $(I_2 - I_3)/I_1 = -(I_3 - I_1)/I_2 = \lambda$, Euler's equations (4) become

$$\frac{d\omega_1}{dt} - \lambda\omega_2\omega_3 = 0, \quad \frac{d\omega_2}{dt} + \lambda\omega_3\omega_1 = 0, \quad \frac{d\omega_3}{dt} = 0. \tag{10}$$

The last of these equations shows that the component of the angular velocity about the axis of revolution of the body is constant. The other two equations give

$$\omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt} = 0,$$

whence

$$\omega_1^2 + \omega_2^2 = \text{const.} = \omega_0^2, \tag{11}$$

where ω_0 denotes the constant angular velocity about the projection of the instantaneous axis on the equatorial plane of the body. The resulting angular velocity is, therefore, constant, viz.

$$\omega = \sqrt{\omega_0^2 + \omega_3^2}. \tag{12}$$

353. The inclination of the instantaneous axis to the principal axes a, b varies, but its inclination to the axis c is constant, viz. $= \cos^{-1}(\omega_3/\omega_0)$. The cone of the body axes is, therefore, a cone of revolution about the axis c, and the polhode is a circle. The herpolhode is, therefore, likewise a circle, and the space axes form a cone of revolution (comp. Art. 347). As the two cones are always in contact along the instantaneous axis, this axis lies in the same plane with the vector $\underset{\cdot}{H}$ and the axis of revolution of the body.

354. To find the angular velocities ω_1, ω_2 as functions of the time, differentiate the first of the equations (10), and eliminate $d\omega_2/dt$ with the aid of the second. This gives

$$\frac{d^2\omega_1}{dt^2} + \lambda^2\omega_3^2\omega_1 = 0,$$

whence

$$\omega_1 = C_1 \cos \lambda\omega_3 t + C_2 \sin \lambda\omega_3 t. \tag{13}$$

The other component, ω_2, can now be found from the first of the equations (10):

$$\omega_2 = \frac{1}{\lambda\omega_3}\frac{d\omega_1}{dt} = -C_1 \sin \lambda\omega_3 t + C_2 \cos \lambda\omega_3 t. \tag{14}$$

To determine the constants C_1, C_2, the initial values of ω_1, ω_2, say at the time $t = 0$, should be known. Let ϵ be the angle made at this time by ω_0 with the principal axis b. Then the initial values of ω_1, ω_2 are $\omega_0 \sin\epsilon$, $\omega_0 \cos\epsilon$, and the substitution of $t = 0$ in (13) and (14) shows that $C_1 = \omega_0 \sin\epsilon$, $C_2 = \omega_0 \cos\epsilon$. Hence we have, finally,

$$\omega_1 = \omega_0 \sin(\lambda\omega_3 t + \epsilon), \quad \omega_2 = \omega_0 \cos(\lambda\omega_3 t + \epsilon), \quad \omega_3 = \text{const.} \tag{15}$$

355. To determine the position of the body at any time t with respect to fixed axes through O, let us take as axis of z the fixed direction of $\underset{\cdot}{H}$, which is perpendicular to the invariable plane. The cosines of the angles made by this axis with the principal axes are found similarly as in Art. 335 (see Fig. 42):

$$\cos ZX' = \sin\phi \sin\theta, \quad \cos ZY' = \cos\phi \sin\theta, \quad \cos ZZ' = \cos\theta.$$

Hence the components $\underset{\cdot}{H}_1 = I_1\omega_1$, $\underset{\cdot}{H}_2 = I_2\omega_2$, $\underset{\cdot}{H}_3 = I_3\omega_3$ of $\underset{\cdot}{H}$ along the principal axes are

$$I_1\omega_1 = \underset{\cdot}{H} \sin\phi \sin\theta, \; I_2\omega_2 = \underset{\cdot}{H} \cos\phi \sin\theta, \; I_3\omega_3 = \underset{\cdot}{H} \cos\theta.$$

These equations give

$$\cos\theta = \frac{I_3\omega_3}{\underset{\cdot}{H}}, \tag{16}$$

and, as $I_1 = I_2$,
$$\tan\phi = \frac{\omega_1}{\omega_2} = \tan(\lambda\omega_3 t + \epsilon),$$

by (15); hence
$$\phi = \lambda\omega_3 t + \phi_0, \tag{17}$$

where ϕ_0 is the value of ϕ for $t = 0$. Thus it appears that the angle θ is constant, while ϕ increases proportionally to the time.

356. To find ψ, we may use the third of the equations (5), Art. 335, viz. $\omega_3 = \dot\phi + \dot\psi \cos\theta$. As $\cos\theta = I_3\omega_3/\underset{\cdot}{H}$, $\dot\phi = \lambda\omega_3$, $\lambda = (I_1 - I_3)/I_1$, we find

$$\frac{d\psi}{dt} = \frac{(1-\lambda)\underset{\cdot}{H}}{I_3} = \frac{\underset{\cdot}{H}}{I_1},$$

whence
$$\psi = \frac{\underset{\cdot}{H}}{I_1} t + \psi_0. \tag{18}$$

It appears then that the equatorial plane $X'Y'$ of the body remains at a constant inclination θ to the invariable plane, while the nodal line ON (Fig. 42) turns uniformly in this invariable plane and a radius of the body in the equatorial plane turns also uniformly in the equatorial plane.

V. *Free Rigid Body.*

I. INITIAL MOTION DUE TO IMPULSES.

357. Kinematically, the most general motion of a rigid body consists, at every instant, of a twist, or screw-motion about a certain line, called the *instantaneous axis* l; that is, the body has, for an element of time, an angular velocity ω about l and at the same time a velocity of translation v along this axis (see Part I., Arts. 43, 44, 294). During the next element of time the body will, in general, rotate about a different axis with a different angular velocity and will have a different linear velocity along the new axis.

358. It has also been shown in kinematics (Part I., Art. 257) that the angular velocity ω about l can be replaced by an equal angular velocity about any parallel axis l', in connection with a certain velocity of translation. For without changing the state of motion we can give the body two equal and opposite rotations about l'; *i.e.* we can introduce along l' (Fig. 44) two equal and opposite rotors $\omega, -\omega$; and $-\omega$ about l' combines with ω about l to a rotor couple, which is equivalent to a velocity of translation $p\omega$, perpendicular to the plane (l, l'), $p = OO'$ being the distance of the parallel axes. The velocity of translation $p\omega$ can now be combined with v to a resultant velocity of translation $v' = \sqrt{v^2 + p^2\omega^2}$ inclined to l' at an angle $\phi = \tan^{-1}(p\omega/v)$.

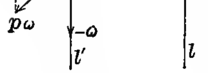

Fig. 44.

It thus appears that the instantaneous motion of a free rigid body can be regarded as a rotation about *any* line parallel to the instantaneous axis, in combination with a certain velocity of translation inclined to this line.

On account of the dynamical properties of the centroid of a rigid body, it will generally be found convenient to select the axis of rotation so as to pass through the centroid; we shall then call it the *centroidal instantaneous axis* $\bar{l}$.

359. Dynamically, the instantaneous motion of a free rigid body is determined by the momenta of its particles. These momenta can be reduced, for any point O as origin, to a resultant momentum and a resultant couple, or angular momentum, and these can be regarded as due to a certain system of impulses. This reduction will at the same time lead to the solution of the converse problem, viz. to determine the initial motion produced by a system of impulses acting on a rigid body at rest, and the change in the instantaneous motion due to such a system when the body is not at rest.

360. Translation. The velocities u of all points being equal and parallel in the case of translation, the momenta mu of all particles are parallel and have (see Arts. 6–8) a single resultant

$$\Sigma mu = Mu,$$

passing through the centroid G of the body. If the whole mass M be regarded as concentrated at the centroid, Mu is the momentum of the centroid. This momentum can be produced by applying at the centroid a single impulse $\underset{\cdot}{R} = Mu$. Hence *to impart to a free rigid body of mass* M *a velocity of translation* u, *it is sufficient to apply at the centroid an impulse* $\underset{\cdot}{\mathrm{R}} = \mathrm{Mu}$.

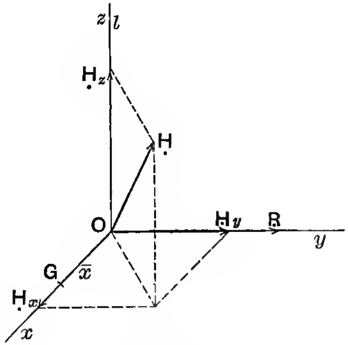

Fig. 45.

361. Rotation. Let us take the instantaneous axis l as axis of z, and the axis of x so as to pass through the centroid G (Fig. 45). The momentum $m\omega r$ of any particle of mass m, at the distance r from l, has the components $-m\omega y$, $m\omega x$, 0; and as $\Sigma my = 0$, $\Sigma mx = M\omega\bar{x}$, the resultant momentum has the components 0, $M\omega\bar{x}$, 0; it is therefore perpendicular to the plane through axis and centroid. *Hence the resultant impulse* $\underset{\cdot}{R}$ *at* O *must be equal in magnitude and direction to the momentum* $M\bar{v}=M\omega\bar{x}$ *of the centroid, due to the rotation* ω *about the instantaneous axis* l.

The resultant angular momentum is found just as in Art. 317; it is $=\omega\sqrt{C^2+D^2+E^2}$ and has the components $-E\omega$ along Ox, $-D\omega$ along Oy, and $C\omega$ along the instantaneous axis Oz.

It follows that *a pure rotation of angular velocity* ω *about an axis* l *can be imparted to a free rigid body by the combined action of an impulse* $\underset{\cdot}{R}$ *and an impulsive couple* $\underset{\cdot}{H}$. The impulse $\underset{\cdot}{R}=M\omega\bar{x}$ is perpendicular to the plane (l, G), and passes through the foot O of the perpendicular let fall from the centroid G on the axis; it vanishes only when $\bar{x}\equiv OG=0$; *i.e.* when the instantaneous axis passes through the centroid. The remarks of Art. 318 apply to the couple without change.

362. As mentioned above, it is often more convenient to take the centroid G as origin for the reduction of the impulses. To reduce the system of impulses $\underset{\cdot}{R}$, $\underset{\cdot}{H}$, determined in the preceding article, to G as origin and to parallel axes (Fig. 46), it is only necessary to apply $\underset{\cdot}{R}$ and $-\underset{\cdot}{R}$ at G; we then have the resultant impulse $\underset{\cdot}{R}=M\omega\bar{x}$ at G, and the couple formed by $\underset{\cdot}{R}$ at O, and $-\underset{\cdot}{R}$ at G. The moment of this couple is $-\underset{\cdot}{R}\bar{x}$ $=-M\omega\bar{x}^2$; its vector is parallel to the instantaneous axis l, and can therefore be added algebraically to $\underset{\cdot}{H}_z$, while $\underset{\cdot}{H}_x$ and $\underset{\cdot}{H}_y$ remain unchanged. Thus the components of the resulting couple for the reduction to the centroid are $\underset{\cdot}{H}_x=-E\omega$, $\underset{\cdot}{H}_y=-D\omega$, $\underset{\cdot}{H}_z=(C-M\bar{x}^2)\omega=C'\omega$, where $C'=C-M\bar{x}^2$ is the moment of inertia about the cen-

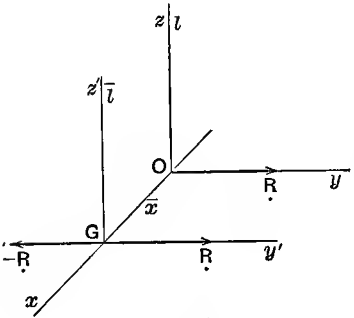

Fig. 46.

troidal axis $\bar{l}$ whose distance from l is $\bar{x}$ (see Art. 250), while D and E are the products of inertia for the new co-ordinate planes through G.

These results can, of course, also be derived directly by reducing the momenta for G as origin, the centroidal instantaneous axis $\bar{l}$ as axis of z, and the plane through G and the instantaneous axis l as the zx-plane.

363. It thus appears that the *form* of the results for this new system of co-ordinates is exactly the same as in Art. 354; but C, D, E refer now to the new co-ordinate axes and planes. Hence *a pure rotation about any instantaneous axis* l, *at the distance* $\bar{x}$ *from the centroid* G, *can be produced by an impulse* $\underset{\cdot}{R}$ *and a couple* $\underset{\cdot}{H}$, *the impulse* $\underset{\cdot}{R}$ *being equal to the momentum* $M\omega\bar{x}$ *of the centroid, due to the rotation, and passing through* G *at right angles to the plane* (l, G), *while the vector of the couple* $\underset{\cdot}{H}$ *has in general three components* $\underset{\cdot}{H}_x = -E\omega$, $\underset{\cdot}{H}_y = -D\omega$, $\underset{\cdot}{H}_z = C\omega$.

The geometrical relation between the vector $\underset{\cdot}{H}$ and the rotor ω can again be illustrated by means of the ellipsoids of inertia, as in Arts. 320, 321. The developments of these articles apply without change if the foot O of the perpendicular let fall from the centroid on the instantaneous axis l_0 be substituted for the fixed point; they apply likewise if the centroid G be substituted for the fixed point, in which case the momental ellipsoid becomes the central, and the reciprocal, the fundamental ellipsoid.

364. The resulting impulse $\underset{\cdot}{R} = M\omega\bar{x}$ vanishes only for $\bar{x} = 0$; *i.e.* when the instantaneous axis l passes through the centroid. In other words, pure rotation about an axis not passing through the centroid cannot be produced by an impulsive couple alone.

On the other hand, pure rotation about a *centroidal* axis can always be regarded as due to an impulsive couple alone; and conversely, *the effect of a single impulsive couple on a free rigid body is to produce pure rotation about a centroidal axis.* But it should always be remembered (see Art. 318) that the axis of rotation l is parallel to the vector $\underset{\cdot}{H}$ of the couple only, and always, if $D=0$ and $E=0$; *i.e.* if the vector $\underset{\cdot}{H}$ is parallel to a principal axis at G. Hence *pure rotation about a centroidal principal axis can be produced by a single couple whose plane is perpendicular to the axis;* and conversely, *a couple whose plane is perpendicular to a centroidal principal axis produces pure rotation about this axis.* The relation between the moment $\underset{\cdot}{H}$ of the couple and the angular velocity ω produced is in this case $\underset{\cdot}{H}=I\omega=M\omega q^2$, where I is the moment, q the radius of inertia for the principal axis.

365. To find the condition under which the system of impulses producing pure rotation may reduce to a single impulse $\underset{\cdot}{R}$, we have only to reduce the system of impulses to its central axis (comp. Part II., Arts. 204–206). For this line which is parallel to $\underset{\cdot}{R}$ has the property that if any point on it be taken as origin of reduction, the couple has its vector parallel to $\underset{\cdot}{R}$ and has its least value $\underset{\cdot}{H}_0$, which is equal to the projection on this line (*i.e.* on the direction of $\underset{\cdot}{R}$) of the vector $\underset{\cdot}{H}$ for *any* reduction. Now, as in our case the components $\underset{\cdot}{H}_x$ and $\underset{\cdot}{H}_z$ are both perpendicular to $\underset{\cdot}{R}$ (see Fig. 45), it follows that

$$\underset{\cdot}{H}_0 = H = -D\omega.$$

This vanishes only with the product of inertia $D=\Sigma myz$.

Hence *pure rotation about an instantaneous axis* l *can be produced by a single impulse* $\underset{\cdot}{\mathrm{R}}=\mathrm{M}\omega\bar{\mathrm{x}}$ *only, and always, if* l *is so situated that the product of inertia* $\mathrm{D}=\Sigma\mathrm{myz}$ *vanishes for the planes through* G *and* l *and through* G *perpendicular to* l. In particular, this is evidently the case *whenever* l *is a principal axis at the foot* O *of the perpendicular let fall on it from the centroid.* (Comp. Arts. 309, 310.)

366. It remains to find the position of the central axis, *i.e.* of the line of action of the single impulse $\underset{\cdot}{R}$ capable of producing pure rotation about the instantaneous axis l provided it satisfies the condition just mentioned. This can be done, if $\underset{\cdot}{R}$ and $\underset{\cdot}{H}$ are known for the centroid, by transferring $\underset{\cdot}{R}$ to parallel positions so as to reduce the components $\underset{\cdot}{H}_z$ and $\underset{\cdot}{H}_x$ to zero. Thus, to destroy $\underset{\cdot}{H}_z = C\omega$ we have only to transfer $\underset{\cdot}{R}$ from G along the axis of x to a point O' (Fig. 47) at a distance $GO' = x_1$ from G such that $\bar{x}x_1 = q^2$, where $\bar{x} = OG$, and q is the radius of inertia for the centroidal instantaneous axis $\bar{l}$. For then the couple resulting from the transfer has a vector along the axis of z equal to $-\underset{\cdot}{R}x_1 = -M\omega\bar{x}x_1 = -M\omega q^2 = -\underset{\cdot}{H}_z$.

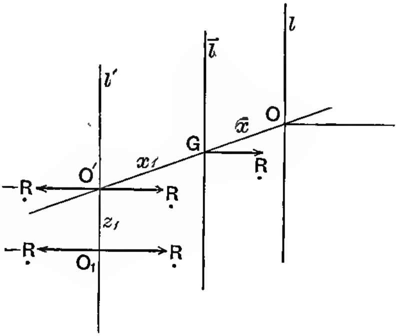

Fig. 47.

Next to destroy $\underset{\cdot}{H}_x = -E\omega = -\omega\Sigma mzx$, we transfer the point of application of $\underset{\cdot}{R}$ parallel to the instantaneous axis l to a point O_1, at a distance $O'O_1 = z_1$ from O', such that $\underset{\cdot}{R}z_1 = -\underset{\cdot}{H}_x$, whence $z_1 = \Sigma mzx / M\bar{x}$.

If the point O_1 be taken as origin of reduction, the system of impulses reduces to $\underset{\cdot}{R}$ at O_1, and the couple $\underset{\cdot}{H}_0 = \underset{\cdot}{H}_y = -D\omega$, whose vector is parallel to $\underset{\cdot}{R}$. Thus *the central axis*, which has of course the direction of $\underset{\cdot}{R}$, and is therefore perpendicular to the plane (l, G), *meets this plane at a point* O$_1$ *whose co-ordinates* are $x_1 = q^2/\bar{x}$, $z_1 = E/m\bar{x}$. It is easy to see that these results agree with the developments of Arts. 309, 310, the centre of percussion, if it exists, being situated on the central axis.

367. Twist or Screw Motion. In the most general case the motion of a rigid body consists of an angular velocity ω about the instantaneous axis l and a simultaneous velocity of translation u along this axis (Art. 357).

Now, by Art. 360, the velocity of translation, u, can be regarded as due to a single impulse $\underset{\cdot}{R}'=Mu$, passing through the centroid G and parallel to u, *i.e.* to l (Fig. 48). Again, by Art. 361, the angular velocity ω about l can be regarded as due to an impulse $\underset{\cdot}{R}''=M\omega\bar{x}$ through G at right angles to the plane (l, G), in combination with a couple whose vector $\underset{\cdot}{H}$ has the components $\underset{\cdot}{H}_x=-E\omega$, $\underset{\cdot}{H}_y=-D\omega$, $\underset{\cdot}{H}_z=C\omega$. The two impulses $\underset{\cdot}{R}'$, $\underset{\cdot}{R}''$ combine to form a single resultant impulse,

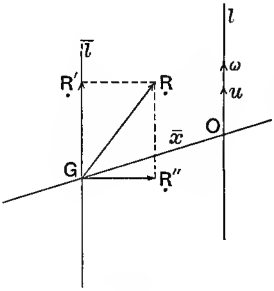

Fig. 48.

$$\underset{\cdot}{R}=\sqrt{\underset{\cdot}{R}'^2+\underset{\cdot}{R}''^2}=M\sqrt{u^2+\omega^2\bar{x}^2},$$

inclined to l at an angle $\phi=\tan^{-1}(\omega\bar{x}/u)$. It should be noticed that the factor $\sqrt{u^2+\omega^2\bar{x}^2}$ is the velocity $\bar{v}$ of the centroid due to the twist, so that *the resultant impulse* $\underset{\cdot}{\mathrm{R}}=\mathrm{M}\bar{\mathrm{v}}$ *is equal to the momentum of the centroid.* The resultant couple $\underset{\cdot}{H}=\omega\sqrt{C^2+D^2+E^2}$ is the same as in the case of pure rotation.

368. The problem of *determining the initial motion produced in a free rigid body at rest by a given system of impulses* finds its geometrical solution in the preceding articles. It should also be remembered that the motion about the centroid takes place as if the centroid were fixed so that all the developments of Arts. 313–323 can be applied by substituting the centroid G for the fixed point O.

It will generally be best to reduce the given impulses to a resultant $\underset{\cdot}{R}$, passing through the centroid, and to a couple $\underset{\cdot}{H}$. By Art. 360, the impulse $\underset{\cdot}{R}$ at G produces a velocity of translation,

$$u=\frac{\underset{\cdot}{R}}{M}. \tag{1}$$

By Arts. 361, 363, 320, 321, the couple $\underset{\cdot}{H}$ produces an angular velocity,

$$\omega=\frac{\underset{\cdot}{H}}{\sqrt{C^2+D^2+E^2}}=\frac{\underset{\cdot}{H}}{M\rho q}=\frac{\underset{\cdot}{H}\rho' q'}{M\epsilon^4}, \tag{2}$$

about a centroidal axis $\bar{l}$ whose direction is conjugate to the plane of the couple $\underset{\cdot}{H}$ in the central ellipsoid, while a plane perpendicular to $\bar{l}$ is conjugate to the direction of the vector $\underset{\cdot}{H}$ in the fundamental ellipsoid. The components of ω along the principal centroidal axes are given by (5), Art. 319, viz.:

$$\omega_1=\frac{\underset{\cdot}{H}_1}{I_1},\quad \omega_2=\frac{\underset{\cdot}{H}_2}{I_2},\quad \omega_3=\frac{\underset{\cdot}{H}_3}{I_3},$$

where I_1, I_2, I_3 are the principal centroidal moments of inertia and $\underset{\cdot}{H}_1$, $\underset{\cdot}{H}_2$, $\underset{\cdot}{H}_3$ the components of $\underset{\cdot}{H}$ along the centroidal principal axes.

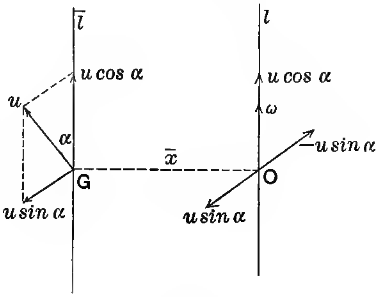

Fig. 49.

The direction of the instantaneous axis having thus been determined, its position can be found by resolving u into a component $u\cos\alpha$ along $\bar{l}$ and a component $u\sin\alpha$ perpendicular to $\bar{l}$ (Fig. 49). The latter component combines with ω about $\bar{l}$ to an equal angular velocity ω about an axis l parallel to $\bar{l}$ at the distance $\bar{x}=-u\sin\alpha/\omega$ from $\bar{l}$.

The initial motion produced by the impulses consists, therefore, of the angular velocity ω about l, and the linear velocity $u\cos\alpha$ along l. These together constitute the resulting twist.

369. Exercises.*

(1) *A homogeneous straight rod* AB = 2a (Fig. 50) *is acted upon by an impulse* $\underset{\cdot}{\mathrm{F}}$, *at the distance* c *from the centroid* G, *at an angle* α *with the rod. Determine the initial motion.*

The reduction to the centroid G gives the impulse $\underset{\cdot}{F}$ at G, which produces a velocity of translation $u=\underset{\cdot}{F}/M$ in the direction of $\underset{\cdot}{F}$, and the couple $\underset{\cdot}{H}$ formed by $\underset{\cdot}{F}$ at C and $-\underset{\cdot}{F}$ at G. The moment of this couple is $Fc\sin\alpha$, and its vector $\underset{\cdot}{H}$ is at right angles to the plane determined by AB and F. As any perpendicular to the rod is a principal axis, the radius of inertia at G being $q=\sqrt{\frac{1}{3}}\,a$, the couple $Fc\sin\alpha$ produces pure rotation about an axis $\bar{l}$ through G at right angles to the plane (AB, F) (see Art. 364), and we have

---

* Most of these problems, as well as the discussions of Arts. 357–368, are adapted from W. SCHELL, *Theorie der Bewegung und der Kräfte*, Vol. II., pp. 352–386.

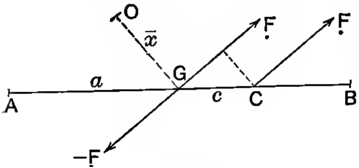

Fig. 50.

$$\omega=\frac{H}{Mq^2}=\frac{3Fc\sin\alpha}{Ma^2}.$$

As the axis $\bar{l}$ of this rotation is perpendicular to the direction of the velocity u of translation, their combined effect is a pure rotation of the same angular velocity ω about a parallel axis l whose position is found as follows (Art. 368): Draw through G, in the plane (AB, F), a perpendicular to F, and on this perpendicular lay off $GO\equiv\bar{x}=u/\omega$. With the values of u and ω given above, we have

$$\bar{x}=\frac{a^2}{3c\sin\alpha},$$

which can easily be constructed geometrically. The parallel to $\bar{l}$ through O is the instantaneous axis about which the rod begins to rotate with the angular velocity ω.

In what direction and at what distance from G must the rod be struck if it is to rotate about a perpendicular through the end A?

(2) *A homogeneous plane lamina of mass* M *receives an impulse* F *in its plane, the distance of the centroid from the direction of* F *being* x_1. *Determine the initial motion.*

The reduction to the centroid G (Fig. 51) gives F at G, and a couple $H=Fx_1$ whose vector is parallel to a principal centroidal axis. The couple produces, therefore, rotation about the perpendicular $\bar{l}$ through G to the plane of the lamina, and this rotation combines with the translation due to F to a single rotation about a parallel axis l. To find the position of l, lay off on the direction of F, drawn through G, a length GK equal to the radius of inertia for $\bar{l}$; join K to the foot O' of the perpendicular let fall from G on F, and draw KO at right angles to KO'. The instantaneous axis l passes through O, since $GO\cdot GO'=q^2$. The angular velocity is $\omega=Fx_1/Mq^2$.

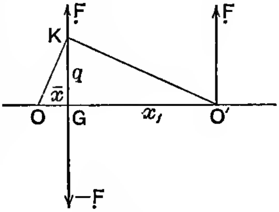

Fig. 51.

(3) *In Ex.* (2), *if the lamina be an ellipse of semi-axes* a, $\frac{1}{2}$a, *at what point of the major axis must it be struck at right angles to this axis in order to rotate initially about a focus?*

(4) *A rigid body of mass* M *receives an impulse* F *parallel to the principal axis* Gy, *and meeting the principal axis* Gx *at the distance* $GO' = x_1$ *from the centroid. Determine the initial motion* (Fig. 52).

The impulse F at O' is equivalent to an equal and parallel impulse F through the centroid G, in combination with the couple $H = Fx_1$ whose vector is parallel to the principal axis Gz. This couple produces, therefore, rotation about the centroidal principal axis Gz, or $\bar{l}$, of angular velocity $\omega = Fx_1/Mq_3^2$, where q_3 is the radius of inertia for $\bar{l}$. The instantaneous axis l is parallel to $\bar{l}$, and meets the axis Gx at a point O such that $GO \cdot GO' = q_3^2$, or putting $GO = \bar{x}$, $GO' = x_1$, such that $\bar{x}x_1 = q_3^2$; it can be constructed as in Ex. (2).

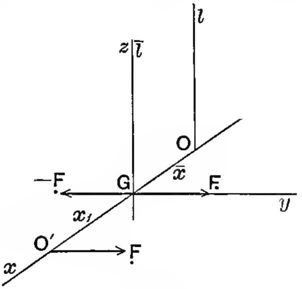

Fig. 52.

(5) *Determine the impulse* F *imparted to a body of mass* M *when struck at the point* O′ (Fig. 53) *of a principal axis* Gx *by a particle of mass* m *moving with a velocity* u *parallel to another principal axis* Gy. *The impact is assumed to be inelastic* (compare Art. 305).

It has been shown in Art. 19 that if a particle of mass m moving with a velocity u impinges upon a particle of mass M at rest, the two particles will, after inelastic impact, move on together with the common velocity $v = mu/(m+M)$. Similarly in our case, as soon as the impact has taken place, the two masses m and M may be regarded as forming a whole, and as moving together. The impulse mu acting on this mass $M+m$ at O' imparts to this point a certain velocity v which can be determined. As the particle m assumes this velocity v after impact, it loses the momentum $mu - mv$, owing to the impact; and this is the impulse

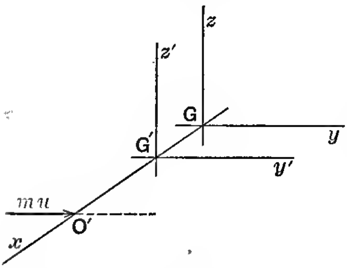

Fig. 53.

$$= m(u - v)$$

of the blow transmitted to the body.

To find v, let G' be the centroid of $M+m$, so that $GO'=x_1$, $GG'=mx_1/(M+m)$, $G'O'=Mx_1/(M+m)$. The principal axes of $M+m$ are $G'x$ and the parallels $G'y'$, $G'z'$ to Gy, Gz. The impulse mu at O' imparts to the mass $M+m$ (see Ex. (4)) a velocity of translation $mu/(M+m)$ parallel to $G'y'$, and an angular velocity $\omega = mu \cdot G'O'/(M+m)q^2$ about $G'z'$, q being the radius of inertia of $M+m$ for $G'z'$. The velocity v of O' is, therefore,

$$v=\frac{mu}{M+m}+\frac{M^2mux_1^2}{(M+m)^3q^2}.$$

For q we have the relation

$$(M+m)q^2=Mq_3^2+M\cdot GG'^2+m\cdot G'O'^2=M\cdot\frac{(M+m)q_3^2+mx_1^2}{M+m},$$

where q_3 is the radius of inertia of M for Gz. Substituting this value of q, we find

$$v=mu\cdot\frac{q_3^2+x_1^2}{(M+m)q_3^2+mx_1^2},$$

and hence

$$F=m(u-v)=mu\cdot\frac{Mq_3^2}{(M+m)q_3^2+mx_1^2}.$$

It thus appears that F equals mu only in the limiting case when $m=0$, $u=\infty$, while $\lim mu=$ const. For given values of m and u, F is a maximum for $x_1=0$; *i.e.* when m strikes the body M at the centroid. In this case $F=mMu/(M+m)$, as it should be, since for direct impact, we have

$$F=m(u-v)=mu-m\cdot mu/(M+m)=mu\cdot M/(M+m).$$

(6) *A free rigid body turns with angular velocity* ω *about an instantaneous axis* l, *which is parallel to a centroidal principal axis and meets another centroidal principal axis at a distance* $\mathrm{GO}=\overline{\mathrm{x}}$ *from the centroid* G (Fig. 54). *A point* P *of the body, situated on the principal axis* GO *at the distance* GP = x *from the centroid, strikes a fixed obstacle; what is the reaction* P *of the obstacle?*

The system of impulses to which the angular velocity ω is due reduces to an impulse $F=Mu=M\omega\bar{x}$ through G, at right angles to the plane (l, G), and a couple $Fx_1=Fq^2/\bar{x}$, where q is the radius of inertia for the centroidal axis $\bar{l}$ parallel to l. The vector of this couple is parallel to l (see Ex. (4)). Just after impact, we have, in addition, the impulsive reaction $\underset{\cdot}{P}$ of the fixed point; hence the resultant impulse $= \underset{\cdot}{F} + \underset{\cdot}{P}$, the resultant couple $= \underset{\cdot}{F}x_1 + \underset{\cdot}{P}x$.

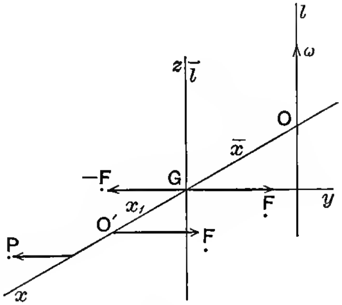

Fig. 54.

As the point P of the body is reduced to rest by the impact, we have only to express the velocity of P and equate it to zero. This gives the condition

$$\frac{\underset{\cdot}{F} + \underset{\cdot}{P}}{M} + \frac{\underset{\cdot}{F}x_1 + \underset{\cdot}{P}p}{Mq^2} \cdot x = 0,$$

whence

$$\underset{\cdot}{P} = -\frac{q^2 + x_1 x}{q^2 + x^2}\underset{\cdot}{F} = -\frac{\bar{x} + x}{q^2 + x^2} \cdot \underset{\cdot}{F}x_1,$$

since $\bar{x}x_1 = q^2$. This becomes $= -\underset{\cdot}{F}$ for $x = 0$ and for $x = x_1$.

Show that there are two points of maximum impact on GO at equal distances from l on opposite sides, and that the maximum impulse is $= -\frac{1}{2}\underset{\cdot}{F}(1 \pm \sqrt{1 + x_1/\bar{x}})$.

(7) *A free rigid body turns with angular velocity* ω *about a centroidal principal axis* $\bar{l}$ *when one of its points* P, *situated at the distance* x *from* $\bar{l}$ *in the centroidal plane perpendicular to* $\bar{l}$, *strikes a fixed obstacle. Determine the impulse on this obstacle, and show that it is greatest when* x = q, *where* q *is the radius of inertia for* $\bar{l}$.

(8) *In Ex.* (6), *determine the initial motion of the body after striking the fixed obstacle.*

2. CONTINUOUS MOTION.

370. In the preceding articles (357–367) it has been shown how to determine a system of impulses capable of producing any given instantaneous state of motion of a free rigid body. Any change in the state of motion can be regarded as due to a system of forces; and by reducing the effective forces of the particles, in a similar way as has been done for the momenta, this system of forces can be determined. This geometrical study of the continuous motion produced by forces is here omitted, as it would require a more complete exposition of the theory of acceleration than has been given in the first part of the present work.

371. Analytically, the continuous motion of a free rigid body is given by the six equations of motion, (4) or (5), Art. 223, and (6) or (7), Art. 224. As pointed out in Art. 233, the motion of the centroid and the motion of the body about the centroid can be considered separately. The former is given by the equations (8), Art. 226, viz.:

$$M\ddot{x}=R_x,\quad M\ddot{y}=R_y,\quad M\ddot{z}=R_z, \tag{1}$$

where M is the mass of the body; $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are the components of the accelerations of the centroid along any three fixed rectangular axes; and R_x, R_y, R_z are the components along the same axes of the resultant R of all the external forces acting on the body.

The motion of the body about the centroid is the same as if the centroid were fixed (Art. 229). It is therefore best studied by taking the centroid G as origin; all the developments of Arts. 324–356 will then apply without change, except that the centroid G must be substituted for the fixed point O. The general equations (3), Art. 326, or Euler's equations (4), Art. 328, can be used to determine the motion about the centroid.

The integration of Euler's equations gives the angular velocities ω_1, ω_2, ω_3 about the three centroidal principal axes of the body. The position of the body, *i.e.* the relation of this system of principal axes to a system of axes through the centroid, parallel to a fixed system, can be determined by means of Euler's angles θ, ϕ, ψ (see Arts. 333–335), or by means of the 9 cosines a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 (Arts. 336, 337).

372. Kinetic Energy. As the instantaneous motion consists of an angular velocity ω about the instantaneous axis l and a velocity of translation u along this axis, the velocity v of any point of the body, at the distance r from l, is $v=\sqrt{u^2+\omega^2r^2}$. Hence the kinetic energy (comp. Art. 235) has the expression

$$T=\Sigma\tfrac{1}{2}mv^2=\Sigma\tfrac{1}{2}m(u^2+\omega^2r^2)=\tfrac{1}{2}Mu^2+\tfrac{1}{2}\omega^2\Sigma mr^2.$$

If q, $\bar{q}$ denote the radii of inertia of the body for the instantaneous axis l and the parallel centroidal axis $\bar{l}$, we have

$$\Sigma mr^2 = Mq^2 = M\bar{q}^2 + M\bar{x}^2,$$

where $\bar{x}$ is the distance between l and $\bar{l}$. Hence denoting by v the velocity of the centroid, $\bar{v} = \sqrt{u^2 + \omega^2\bar{x}^2}$, we find

$$T = \tfrac{1}{2}M(u^2 + \omega^2\bar{x}^2) + \tfrac{1}{2}M\omega^2\bar{q}^2 = \tfrac{1}{2}M\bar{v}^2 + \tfrac{1}{2}M\bar{q}^2\omega^2. \qquad (2)$$

It thus appears that the kinetic energy consists of two parts, $T = T_1 + T_2$, one of which,

$$T_1 = \tfrac{1}{2}M\bar{v}^2,$$

may be called the *kinetic energy of the centroid* (the whole mass M being regarded as concentrated at the centroid), while the other,

$$T_2 = \tfrac{1}{2}M\bar{q}^2\omega^2,$$

the so-called *kinetic energy of rotation*, is the kinetic energy which the body would possess if it were rotating with the angular velocity ω, not about the instantaneous axis l, but about the parallel centroidal axis $\bar{l}$. The developments of Arts. 322 and 323 apply without change to T_2.

373. Numerous exercises and applications will be found in the works of Price, Besant, Williamson and Tarleton, Walton, quoted in Art. 159; but above all in E. J. ROUTH, *Dynamics of a system of rigid bodies*, Elementary part, fifth edition, 1891; Advanced part, fourth edition, 1884; London and New York, Macmillan. Illustrations and examples, as well as further developments of the theory, will also be found in the works of Schell and Budde (Art. 159); in the French collections of problems by M. Jullien and by A. de Saint-Germain; in J. PETERSEN, *Lehrbuch der Dynamik fester Körper*, deutsch von R. von Fischer-Benzon, Kopenhagen, Höst, 1887; E. BOUR, *Cours de mécanique et machines*, III^e fascicule, Paris, Gauthier-Villars, 1874; and the original memoirs of L. POINSOT, in particular his *Théorie nouvelle de la rotation des corps*, Paris, Bachelier, 1852 (also in Liouville's *Journal de mathématiques*, Vol. XVI.), and his *Précession des équinoxes*, Paris, Mallet-Bachelier, 1857. Among the numerous French treatises on theoretical mechanics those of Poisson, Sturm, Resal, Collignon deserve especially to be mentioned here.

CHAPTER VII.

MOTION OF A VARIABLE SYSTEM.

374. In the present chapter we shall consider very briefly the motion of a general system of n particles, connected in any way whatever, subject to any conditions or constraints, and acted upon by any forces.

The forces can be distinguished as *external* and *internal.* The latter are exerted by certain particles, or groups of particles, of the system on other particles of the same system, while the former proceed from without the system. Thus, in considering our solar system, the attractions between its various members are internal forces, while the attractions of the fixed stars on the sun or planets would represent external forces.

Besides these two kinds of forces there may be forces replacing constraints, such as reactions of fixed points, lines, or surfaces, friction, etc.

I. *Free System.*

375. If the system be free, *i.e.* if it be only acted upon by external and internal forces, while there are no constraints or conditions prescribed for it, the establishment of the general equations of motion is simple, although their integration generally presents insuperable difficulties. The problem of two bodies (Arts. 150–158) is the simplest special case.

The general principles laid down in Arts. 218–238 for the motion of a rigid body apply almost without change to a free system of particles; indeed, they apply even to a constrained system, provided that all conditions and constraints are replaced by forces and that these constraining forces are included among the forces X, Y, Z, acting on the particles. Thus in particular, the general equations of motion of a rigid body, viz. (4) or (5), Art. 223, and (6) or (7), Art. 224, hold for a variable system. For they express the necessary, though not in general sufficient, conditions of equilibrium of the forces acting on the particles with the reversed effective forces of these particles; and this equilibrium is not changed by making the distances between the particles invariable; *i.e.* by what is sometimes called *solidifying* the system. But it should be observed that the reductions of the systems of momenta and effective forces, given in the chapter on the rigid body, do not in general hold for a variable system.

376. Let F be the resultant of all the external and internal forces acting on one of the n particles; X, Y, Z its components along a system of fixed rectangular axes; x, y, z the co-ordinates of the particle, and m its mass. Just as in Arts. 219, 220, we have the equations of motion of the particle

$$m\ddot{x}=X,\quad m\ddot{y}=Y,\quad m\ddot{z}=Z. \qquad (1)$$

There are 3 such equations for each particle, and hence $3n$ for the whole system. These $3n$ equations express the equilibrium of the system of forces composed of the external, internal, and reversed effective forces.

377. Applying the principle of virtual work to this system of forces, we find *d'Alembert's equation*

$$\Sigma(-m\ddot{x}+X)\delta x+\Sigma(-m\ddot{y}+Y)\delta y+\Sigma(-m\ddot{z}+Z)\delta z=0, \qquad (2)$$

in which δx, δy, δz are the components of an arbitrary displacement δs of the particle m. As there are $3n$ such arbitrary component displacements, the equation (2) is equivalent to the $3n$ equations (1).

If written in the form

$$\Sigma m(\ddot{x}\delta x+\ddot{y}\delta y+\ddot{z}\delta z)=\Sigma(X\delta x+Y\delta y+Z\delta z), \qquad (3)$$

it expresses the equality of the sum of the virtual works of the effective forces to the sum of the virtual works of the external and internal forces, for any infinitesimal displacement of the system. The internal forces do not enter into this equation if they occur in pairs of equal and opposite forces, as will usually be the case.

378. As there are no conditions, we may select for δs the actual displacement ds of every particle, so that the equation (3) becomes

$$\Sigma m(\ddot{x}dx+\ddot{y}dy+\ddot{z}dz)=\Sigma(Xdx+Ydy+Zdz).$$

The left-hand member is the exact differential $\Sigma\frac{1}{2}md(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ $=d\Sigma\frac{1}{2}mv^2$. Hence, integrating between the limits 0 and t, and denoting by v_0 the velocity of the particle m at the time 0, we find

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0^2=\Sigma\int_0^t(Xdx+Ydy+Zdz). \tag{4}$$

This is the *equation of kinetic energy*. The right-hand member represents the work done by the forces during the time t.

379. If there exists a force function or potential U for the forces X, Y, Z, *i.e.* if these forces are the partial derivatives with respect to x, y, z of one and the same function U, the system is said to be *conservative*. We have then

$$\Sigma(Xdx+Ydy+Zdz)=dU,$$

and the integration of (4) gives

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0^2=U-U_0, \tag{5}$$

where U_0 is the value of U for $t=0$.

Denoting as usual the kinetic energy by T, the potential energy $-U$ by V, this equation can be written

$$T+V=T_0+V_0=\text{const.}; \tag{6}$$

it expresses the **principle of the conservation of energy**. (Comp. Art. 79).

380. Exercises. Show the existence of a force-function and find its expression in the following cases (comp. Art. 86):

(1) When the resulting force F at each particle m is constant in magnitude and direction (gravity).

(2) When the forces F are all attractions, each being directed to some fixed centre O and a function of the distance r from this centre.

(3) When the forces F are the mutual attractions of the particles constituting the system.

381. A variable system of n particles possesses a centroid whose co-ordinates $\bar{x}, \bar{y}, \bar{z}$ satisfy the equations

$$M \cdot \bar{x} = \Sigma mx, \quad M \cdot \bar{y} = \Sigma my, \quad M \cdot \bar{z} = \Sigma mz.$$

The developments of Arts. 226, 227, in particular *the principle of the conservation of linear momentum*, or the principle of the conservation of the motion of the centroid, hold for a variable system just as well as for a rigid body. The position of the centroid in the system is of course variable with the time.

The principle just referred to asserts that, *if* $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$, *the centroid of the system is at rest, or moves with constant velocity in a straight line.* It should be noticed that the conditions $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$ do not mean that there are no forces at all acting on the system; they only mean that the resultant of these forces reduces to zero while there may be a resulting couple different from zero. The principle would, for instance, apply to the solar system if the action of the fixed stars be regarded as vanishing or as reducing to a couple; the mutual attractions of the various members of the system occur in pairs of equal and opposite forces, and have, therefore, a resultant zero.

382. Similarly, the developments of Arts. 228–232, in particular *the principle of the conservation of angular momentum*, or of areas, and the properties of the invariable plane, apply without change to the free variable system.

II. *System Subject to Conditions.*

383. The constraints and conditions to which a variable system is subject may be of very different kinds. In general, however, they can be imagined as replaced by certain forces, called *constraining forces* or *reactions*, by the introduction of which the system becomes free. On the other hand, it may be noticed that internal forces, such as tensions of connecting rods or strings, can sometimes be regarded as constraining conditions.

If all conditions and constraints be expressed by means of forces, and these forces be included among the forces X, Y, Z, the equations of motion of the particle m have again the form (1), Art. 376, and the principle of virtual work gives the equation of d'Alembert (2), Art. 377. But it should be noticed that, in general, the constraints will do no work if the displacements δx, δy, δz are properly selected; in other words, if the displacements be taken so as to be compatible with the conditions to which the system is subject, the constraining forces will not enter into the equation (2). This is **d'Alembert's principle.**

384. Before further developing this idea it may be well to indicate here the considerations by which d'Alembert himself (and, in more exact language, Poisson) explained his celebrated principle.

Any particle m of the system is acted upon at any time t by two kinds of forces, the given external and internal forces, whose resultant we denote by F, and the internal reactions and constraining forces whose resultant we call F' (Fig. 55). The resultant of F and F' must be geometrically equal to the effective force mj, where j is the acceleration of the particle at the time t.

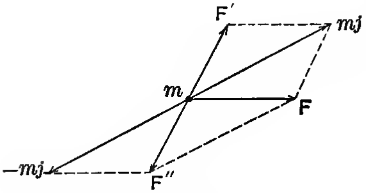

Fig. 55.

Now, if we introduce at m the equal and opposite forces mj, $-mj$, the action of F and F', and hence the motion of the particle, will not be changed. But we can now combine F and $-mj$ to a resultant F''. Since $F, F', -mj$ are in equilibrium, the forces F' and F'' are in equilibrium; *i.e.* F'' is equal and opposite to F', as appears from the figure.

The figure also shows that F can be resolved into the components mj and F''; the former component, mj, produces the actual motion of the particle, while the latter, F'', is consumed in overcoming the internal reactions and constraints represented by F'. This component F'' of F is therefore called by d'Alembert the *lost force*. As $F'+F''=0$ at every particle of the system, d'Alembert's principle can be expressed by saying that, at every moment during the motion, *the lost forces are in equilibrium with the constraints of the system.*

If the constraints, instead of being expressed by means of forces, are given by equations of condition, we may express the same idea by saying that, *owing to the given conditions, the lost forces form a system in equilibrium.*

385. We shall now assume that the constraints or conditions to which the system is subject are expressed by means of equations (the case of conditions expressed by inequalities is excluded) between the co-ordinates x, y, z of the particles and the time t. In the most general case these equations might also contain the velocities of the particles; this case, however, will not be considered here.

Let there be k conditions

$$\phi(t, x_1, y_1, z_1, x_2, \cdots)=0, \quad \psi(t, x_1, y_1, z_1, x_2, \cdots), \cdots \qquad (1)$$

for a system of n points. Then the number of the independent equations of motion will be $3n-k$. For these equations must express the equilibrium of the given forces, together with the reversed effective forces, under the given conditions; and for this equilibrium it is sufficient that *the virtual work should vanish for any displacement compatible with the conditions*, the work of the reactions and constraining forces being zero for such virtual displacement. In other words, in d'Alembert's equation

$$\Sigma(-m\ddot{x}+X)\,\delta x+\Sigma(-m\ddot{y}+Y)\,\delta y+\Sigma(-m\ddot{z}+Z)\,\delta z=0, \quad (2)$$

the constraining forces due to the conditions will not appear if the displacements δx, δy, δz be so selected as to be compatible with the k conditions (1).

Now this will be the case if these displacements be made to satisfy the equations that result from differentiating the conditions (1), viz.

$$\begin{aligned}\Sigma(\phi_x\delta x+\phi_y\delta y+\phi_z\delta z)&=0,\\ \Sigma(\psi_x\delta x+\psi_y\delta y+\psi_z\delta z)&=0, \quad (3)\\ \ldots\ldots\ldots\ldots\end{aligned}$$

It should be noticed that in this differentiation, or rather variation, the time t is regarded as constant. If, for instance, one of the conditions (1) constrain a particle to a curve or surface varying with the time, say the surface of the moving earth, or that of a projectile in motion, the displacement is called *virtual*, or compatible with the condition, if it takes place on the curve or surface regarded momentarily as fixed (comp. Art. 193). Indeed, when the conditions contain the time, the statement that a virtual displacement is one compatible with the conditions has no definite meaning; virtual displacements are then *defined* as displacements satisfying the equations (3).

386. The k equations (3) make it possible to eliminate k of the $3n$ displacements from d'Alembert's equation (2). There will remain $3n-k$ independent arbitrary displacements, whose coefficients equated to zero give the $3n-k$ equations of motion.

Applying the method of indeterminate multipliers (comp. Art. 194) to perform this elimination in a systematic way, we have to multiply the k equations (1) by indeterminate factors λ, μ, $\cdots$, and to add them to equation (2). The k multipliers λ, μ, $\cdots$ can then be so selected as to make the coefficients of k of the $3n$ displacements δx, δy, δz vanish. As the remaining $3n-k$

displacements are arbitrary, their coefficients must also vanish separately. Thus it follows that the coefficients of *all* the displacements in the resulting equation must vanish, and we have n sets of 3 equations of the type

$$\begin{aligned} m\ddot{x} &= X+\lambda\phi_x+\mu\psi_x+\cdots,\\ m\ddot{y} &= Y+\lambda\phi_y+\mu\psi_y+\cdots,\\ m\ddot{z} &= Z+\lambda\phi_z+\mu\psi_z+\cdots. \end{aligned} \tag{4}$$

It is apparent from these equations that the constraining force acting on the particle m has the components

$$\begin{aligned} X' &= \lambda\phi_x+\mu\psi_x+\cdots,\\ Y' &= \lambda\phi_y+\mu\psi_y+\cdots,\\ Z' &= \lambda\phi_z+\mu\psi_z+\cdots. \end{aligned} \tag{5}$$

387. It has thus been shown that a system of n particles subject to k conditions has $3n-k$ independent equations of motion. The equations can be obtained either by eliminating from d'Alembert's equation (2) k of the $3n$ displacements δx, δy, δz by means of the k equations (3), and then equating to zero the coefficients of the remaining $3n-k$ arbitrary displacements, or they may be regarded as represented by the $3n$ equations (4), since these equations contain k arbitrary quantities λ, μ, $\cdots$. In this latter form they are sometimes denoted as *Lagrange's first form of the equations of motion.*

388. It follows from the remark at the end of Art. 385, that the actual displacements dx, dy, dz of the particles can be selected as virtual displacements only, and always, when the conditional equations (1) do not contain the time. If this condition be fulfilled, d'Alembert's equation (2) can be written

$$\Sigma m(\ddot{x}dx+\ddot{y}dy+\ddot{z}dz)=\Sigma(Xdx+Ydy+Zdz),$$

or

$$d\Sigma\tfrac{1}{2}mv^2=\Sigma(Xdx+Ydy+Zdz). \tag{6}$$

This relation can also be deduced from the equations (4) by multiplying them by $\dot{x}dt$, $\dot{y}dt$, $\dot{z}dt$, and summing the equations for all the particles. The left-hand member of the resulting equation is again $d\Sigma\frac{1}{2}mv^2$. In the right-hand member we find, besides the term $\Sigma(Xdx+Ydy+Zdz)$, such terms as

$$\lambda\Sigma(\phi_x\dot{x}+\phi_y\dot{y}+\phi_z\dot{z})dt,\ \mu\Sigma(\psi_x\dot{x}+\psi_y\dot{y}+\psi_z\dot{z})dt, \cdots.$$

The coefficients of λ, μ, $\cdots$ vanish only when the conditions (1) are independent of the time, for then the differentiation of these equations (1) gives

$$\Sigma(\phi_x\dot{x}+\phi_y\dot{y}+\phi_z\dot{z})=0,\ \Sigma(\psi_x\dot{x}+\psi_y\dot{y}+\psi_z\dot{z})=0, \cdots.$$

In other words, in this case the constraining forces do no work during the actual displacement of the system, as they are all perpendicular to the paths of the particles, and we find equation (6).

If, however, the conditional equations (1) contain the time, their differentiation gives

$$\Sigma(\phi_x\dot{x}+\phi_y\dot{y}+\phi_z\dot{z})+\phi_t=0,\quad \Sigma(\psi_x\dot{x}+\psi_y\dot{y}+\psi_z\dot{z})+\psi_t=0,\ \cdots,$$

and we find in the place of equation (6):

$$d\Sigma\tfrac{1}{2}mv^2=\Sigma(Xdx+Ydy+Zdz)-\lambda\phi_t-\mu\psi_t\cdots. \tag{7}$$

389. If the conditional equations do not contain the time, and if, moreover, there exists a force-function U for all the forces, equation (6) can be put into the form

$$d\Sigma\tfrac{1}{2}mv^2=dU,$$

which gives, by integration,

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0^2=U-U_0, \tag{8}$$

or, by putting $U=-V$,

$$T+V=T_0+V_0. \tag{9}$$

This equation expresses the *principle of the conservation of energy.*

It should be noticed that, even when there exists no force-function, the elementary work $\Sigma(Xdx+Ydy+Zdz)$ is a quantity independent of the co-ordinate system, and the sum of these elementary works for a finite time, say from $t=0$ to $t=t$, represents a certain finite amount of work $W=\int_0^t \Sigma(Xdx+Ydy+Zdz)$, so that equation (6) gives always

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0^2=W.$$

This means that *if the conditions are independent of the time, the increase of kinetic energy during any interval of time is equal to the work done during this time by all the external and internal forces.*

But when a force-function exists, this work is $W=U-U_0$, where U is a function of the co-ordinates only. The work done by the forces depends therefore only on the initial and final values of these co-ordinates; *i.e.* on the initial and final configuration of the system, but not on the character of the motion by which the system is brought from the initial to the final position.

390. It has been shown in Art. 222 that, for an *invariable* system of n points, *i.e.* for a free rigid body, the number of conditions is $k=3n-6$; hence the number of independent equations of motions of a free rigid body is $3n-(3n-6)=6$.

A rigid body with a fixed axis (Art. 291) has but one degree of freedom and 5 constraints; *i.e.* its position is given by a single variable, say the angle of rotation, θ, about the fixed axis. The motion of such a body is therefore given by a single equation.

A rigid body that can turn about and also slide along a fixed axis has 4 constraints and 2 degrees of freedom; it has therefore 2 equations of motion, and 2 variables are sufficient to determine any particular position of the body, say the angle θ and the distance x measured along the axis of rotation.

A rigid body with one fixed point (Art. 311) is an example of an invariable system with 3 constraints and 3 degrees of freedom. Three variables are necessary and sufficient to determine a particular position, and the number of independent equations of motion is 3.

Similarly, it will be seen in every other case that a rigid body has as many independent equations of motion as it has degrees of freedom, or as it requires variables to fix its position. These variables may be called the *co-ordinates of the rigid body.* Thus a free rigid body has 6 co-ordinates corresponding to its 6 degrees of freedom and 6 equations of motion; we might take as such co-ordinates the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centroid and Euler's angles θ, ϕ, ψ.

391. These considerations can be generalized so as to apply to a general variable system of n points with k conditions. Such a system is said to have $3n-k=m$ co-ordinates because it has $3n-k=m$ independent equations of motion (Art. 385). In other words, in the place of the $3n$ Cartesian co-ordinates x, y, z of the n points, subject to k conditional equations, we may introduce $3n-k=m$ independent variables, say q_1, q_2, $\cdots$ q_m, which are so selected as to satisfy the k conditions (1) identically. These variables are called the **Lagrangian**, or **generalized, co-ordinates** of the system.

By the introduction of these new variables the equations of motion (4) assume a form which is known as the *second Lagrangian form.*

Suppose, for instance, that the system is subject to only one condition, viz. that one point P_1 of the system should remain on the surface of the ellipsoid

$$\phi \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1=0.$$

If we select two new variables q_1, q_2 connected with x_1, y_1, z_1 by the equations $x_1=a\cos q_1$, $y_1=b\sin q_1\cos q_2$, $z_1=c\sin q_1\sin q_2$, the condition $\phi=0$ is satisfied identically in the new co-ordinates q_1, q_2. Hence, by introducing q_1, q_2 in the place of x_1, y_1, z_1, the condition $\phi=0$ is eliminated from the problem.

We now proceed to establish the equations of motion in the second Lagrangian form, for a variable system of n points with the k conditions (1), *i.e.* to introduce $3n-k=m$ new variables or generalized co-ordinates, $q_1, q_2, \cdots q_m$ in the place of the $3n$ Cartesian co-ordinates $x_1, y_1, z_1, x_2, \cdots z_n$, selecting the new co-ordinates so as to satisfy the conditions (1) identically (comp. Arts. 210–216).

392. The Cartesian co-ordinates x, y, z of any one of the n points, as well as their time derivatives $\dot{x}, \dot{y}, \dot{z}$, are functions of $q_1, q_2, \cdots q_m$ and of the time t. We have therefore

$$\dot{x}=\frac{\partial x}{\partial t}+\frac{\partial x}{\partial q_1}\dot{q}_1+\frac{\partial x}{\partial q_2}\dot{q}_2+\cdots+\frac{\partial x}{\partial q_m}\dot{q}_m, \tag{10}$$

with similar expressions for $\dot{y}$ and $\dot{z}$. Thus $\dot{x}$, $\dot{y}$, $\dot{z}$ are represented as functions of the independent variables $t, q_1, q_2, \cdots q_m, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_m$.

Differentiating $\dot{x}$ partially with respect to any one, q, of the quantities $q_1, q_2, \cdots q_m$, we find

$$\frac{\partial \dot{x}}{\partial q}=\frac{\partial^2 x}{\partial q \partial t}+\frac{\partial^2 x}{\partial q \partial q_1}\dot{q}_1+\frac{\partial^2 x}{\partial q \partial q_2}\dot{q}_2+\cdots+\frac{\partial^2 x}{\partial q \partial q_m}\dot{q}_m$$

$$=\frac{\partial}{\partial t}\frac{\partial x}{\partial q}+\frac{\partial}{\partial q_1}\frac{\partial x}{\partial q}\cdot\dot{q}_1+\frac{\partial}{\partial q_2}\frac{\partial x}{\partial q}\cdot\dot{q}_2+\cdots+\frac{\partial}{\partial q_m}\frac{\partial x}{\partial q}\cdot\dot{q}_m.$$

We have therefore,

$$\frac{\partial \dot{x}}{\partial q}=\frac{d}{dt}\frac{\partial x}{\partial q},\quad \frac{\partial \dot{y}}{\partial q}=\frac{d}{dt}\frac{\partial y}{\partial q},\quad \frac{\partial \dot{z}}{\partial q}=\frac{d}{dt}\frac{\partial z}{\partial q}. \tag{11}$$

Again, differentiating (10) partially with respect to $\dot{q}$, we have

$$\frac{\partial \dot{x}}{\partial \dot{q}}=\frac{\partial x}{\partial q},\quad \frac{\partial \dot{y}}{\partial \dot{q}}=\frac{\partial y}{\partial q},\quad \frac{\partial \dot{z}}{\partial \dot{q}}=\frac{\partial z}{\partial q}. \tag{12}$$

Let us also form the derivatives of the kinetic energy

$$T=\Sigma\tfrac{1}{2}\,m(\dot{x}^2+\dot{y}^2+\dot{z}^2), \tag{13}$$

viz. $$\frac{\partial T}{\partial q}=\Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial q}+\dot{y}\frac{\partial \dot{y}}{\partial q}+\dot{z}\frac{\partial \dot{z}}{\partial q}\right),$$

which by (11) and (12) becomes

$$\frac{\partial T}{\partial q}=\Sigma m\left(\dot{x}\frac{d}{dt}\frac{\partial \dot{x}}{\partial \dot{q}}+\dot{y}\frac{d}{dt}\frac{\partial \dot{y}}{\partial \dot{q}}+\dot{z}\frac{d}{dt}\frac{\partial \dot{z}}{\partial \dot{q}}\right); \tag{14}$$

and
$$\frac{\partial T}{\partial \dot q}=\Sigma m\left(\dot x\frac{\partial \dot x}{\partial \dot q}+\dot y\frac{\partial \dot y}{\partial \dot q}+\dot z\frac{\partial \dot z}{\partial \dot q}\right). \qquad (15)$$

From (15) and (14) we find

$$\frac{d}{dt}\,\frac{\partial T}{\partial \dot q}=\Sigma m\left(\ddot x\frac{\partial \dot x}{\partial \dot q}+\ddot y\frac{\partial \dot y}{\partial \dot q}+\ddot z\frac{\partial \dot z}{\partial \dot q}\right)+\frac{\partial T}{\partial q}. \qquad (16)$$

393. Thus prepared we can introduce the new co-ordinates q into the equations of motion (4) by multiplying these equations by $\partial x/\partial q$, $\partial y/\partial q$, $\partial z/\partial q$, and adding them throughout the whole system; this gives:

$$\Sigma m\left(\ddot x\frac{\partial x}{\partial q}+\ddot y\frac{\partial y}{\partial q}+\ddot z\frac{\partial z}{\partial q}\right)=\Sigma\left(X\frac{\partial x}{\partial q}+Y\frac{\partial y}{\partial q}+Z\frac{\partial z}{\partial q}\right); \qquad (17)$$

the coefficients of λ, μ, ⋯ all disappear in the summation, since, by hypothesis, the new co-ordinates satisfy the conditional equations (1) identically.

The right-hand member of (17) we shall denote by Q (comp. Arts. 180, 211); the left-hand member can be put into a more convenient form by means of (16) and (12). Thus we find finally *the equations of motion in the second Lagrangian form:*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q}-\frac{\partial T}{\partial q}=Q. \qquad (18)$$

As there is one such equation for every one of the Lagrangian co-ordinates q_1, q_2, ⋯ q_m, the number of such equations is $m=3n-k$. They are obtained from the type (18) by attaching successively the subscripts 1, 2, ⋯ m to each of the symbols q, $\dot q$, Q.

394. In the particular case of a *conservative system*, *i.e.* when there exists a force function U such that

$$\Sigma X=\frac{\partial U}{\partial x},\quad \Sigma Y=\frac{\partial U}{\partial y},\quad \Sigma Z=\frac{\partial U}{\partial z},$$

the quantity Q in (18) is evidently $=\partial U/\partial q$, so that the equations of motion assume the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q}=\frac{\partial}{\partial q}(T+U). \qquad (19)$$

This equation can be derived more directly from equation (16) by considering an infinitesimal displacement of the system. If δq be the change of the co-ordinate q in such a displacement, the partial changes, or variations, of x, y, z will be

$$\frac{\partial x}{\partial q}\delta q,\quad \frac{\partial y}{\partial q}\delta q,\quad \frac{\partial z}{\partial q}\delta q\,;$$

hence the work of the effective forces $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$, for the whole system, is

$$\Sigma m\delta q\left(\ddot{x}\frac{\partial x}{\partial q}+\ddot{y}\frac{\partial y}{\partial q}+\ddot{z}\frac{\partial z}{\partial q}\right)=\delta q\Sigma m\left(\ddot{x}\frac{\partial \dot{x}}{\partial \dot{q}}+\ddot{y}\frac{\partial \dot{y}}{\partial \dot{q}}+\ddot{z}\frac{\partial \dot{z}}{\partial \dot{q}}\right).$$

This is the amount by which the *potential energy* $V=-U$ is diminished; it is, therefore, equal to $(\partial U/\partial q)\delta q$. Hence the second term in the right-hand member of (16) can be replaced by $\partial U/\partial q$; this at once gives equation (19).

395. From Lagrange's equations it is easy to derive **Hamilton's principle.**

Let each of the equations (18) be multiplied by the infinitesimal displacement, or variation, δq; let the equations be added, multiplied by dt, and integrated from t_1 to t_2:

$$\int_{t_1}^{t_2}\Sigma\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q}-Q\right)\delta q dt=0. \tag{20}$$

The first term can be transformed by partial integration; remembering that $d(\delta q)/dt=\delta(dq)/dt$, we have

$$\int_{t_1}^{t_2}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}\delta q dt=\left(\frac{\partial T}{\partial \dot{q}}\delta q\right)_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{\partial T}{\partial \dot{q}}\delta \dot{q} dt.$$

If now the variations δq be so selected as to vanish both at the time t_1 and at the time t_2, the first term vanishes at both limits. Hence equation (20) assumes the form

$$\int_{t_1}^{t_2}\Sigma\left(\frac{\partial T}{\partial \dot{q}}\delta \dot{q}+\frac{\partial T}{\partial q}\delta q+Q\delta q\right)dt=0.$$

As $\Sigma\left(\frac{\partial T}{\partial \dot{q}}\delta \dot{q}+\frac{\partial T}{\partial q}\delta q\right)=\delta T$ and $\Sigma Q\delta q=\delta U$ for a conservative system, and $=\delta W$ for a general system (Art. 389), the equation reduces to the simple form

$$\int_{t_1}^{t_2}(\delta T+\delta W)dt=0, \qquad (21)$$

in the general case, and to

$$\delta\int_{t_1}^{t_2}(T+U)dt=0, \text{ or } \delta\int_{t_1}^{t_2}(T-V)dt=0, \qquad (22)$$

in the case of a conservative system.

396. Hamilton's principle consists in the proposition that the equation (21) or (22) holds for any displacements of the system compatible with the conditions (1), provided these displacements be zero at the times t_1 and t_2. Assuming the existence of a force-function, *i.e.* taking (22) as the expression of Hamilton's principle, its meaning can be expressed as follows. If we consider any two positions of the moving system, say the positions which it occupies at the times t_1 and t_2, the motion by which the system actually passes from the former to the latter position is characterized, and distinguished from any other imaginable ways of passing from the former to the latter, by the property that the variation of the time-integral of the difference between kinetic and potential energy vanishes. In other words, *for the actual motion the average value of the difference between kinetic and potential energy during any time is a minimum.*

The chief advantage of Hamilton's principle lies in the fact that it is independent of any co-ordinate system, and can therefore be used as a convenient starting-point for introducing the variables best adapted to the needs of the particular problem.

397. A more complete discussion of Lagrange's equations of motion and of Hamilton's as well as other similar principles of dynamics, such as the principle of least action, of least constraint, etc., will be found in the work of E. J. Routh (see Art. 373) and in W. Schell's *Theorie der Bewegung und der Kräfte*, Vol. II., pp. 544–571. The kinetics of the variable system forms the basis for the investigations of mathematical physics, *i.e.* for the theory of elastic bodies, of fluid and liquid motion, of heat, light, electricity, and magnetism. The following works are particularly recommended as introductions to this subject: THOMSON AND TAIT, *Natural Philosophy*, Vol. I., parts 1 and 2, Cambridge, University Press (New York, Macmillan), new edition, 1890; W. VOIGT, *Elementare Mechanik als Einleitung in das Studium der theoretischen Physik*, Leipzig, Veit, 1889; G. KIRCHHOFF, *Vorlesungen über mathematische Physik;* Mechanik, 3te Auflage, Leipzig, Teubner, 1883.

ANSWERS.

Pages 9, 10.

(1) 40.

(2) (*a*) $4\frac{1}{7}$ and $5\frac{1}{7}$; (*b*) $-2\frac{5}{7}$, $6\frac{2}{7}$.

(3) If the original velocities are of the same sense, $v = 42\frac{4}{9}$, $v' = 46\frac{1}{9}$; if not, $v = -19\frac{7}{9}$, $v' = 10\frac{5}{9}$.

(4) $e = 0$ gives (*a*) $44\frac{7}{12}$, (*b*) $-2\frac{1}{12}$; $e = 1$ gives (*a*) $v = 38\frac{1}{6}$, $v' = 49\frac{1}{6}$, (*b*) $v = -55\frac{1}{6}$, $v' = 35\frac{5}{6}$.

(5) $v = -eu$.

(7) $8\frac{1}{3}$ ft.

(8) (*a*) 0·31 ft.; (*b*) $9\frac{1}{2}$ sec.; (*c*) $66\frac{6}{11}$ ft.

(10) $\left(\frac{1+e}{2}\right)^n u$.

(11) (*a*) 4 ft. per second; (*b*) $17\frac{1}{3}$ ft. per second.

(12) For $e = 0$: (*a*) $v = \frac{m}{m+m'}u$; (*b*) $\lim v = 0$; (*c*) $\lim v = u'$. For $e = 1$: (*a*) $v = \frac{m-m'}{m+m'}u$, $v' = \frac{2m}{m+m'}u$; (*b*) $\lim v = -u$, $\lim v' = 0$; (*c*) $\lim v = 2u' - u$, $\lim v' = u'$. Interpret these results.

Page 14.

(1) About 450 pounds; 9·375 and 0·191 foot-pounds.

(2) 156·85 foot-poundals.

(3) $16\frac{1}{2}$ foot-tons; 9 miles per hour.

(6) $2\frac{1}{6}$ tons.

(7) 13 and 2 foot-tons.

Page 16.

(1) 56·83 F.P.S. units.

(2) 16 ft. per second.

(3) $v = 10$, $\beta = 48\frac{1}{3}^\circ$, $v' = 16\frac{1}{2}$, $\beta' = 15\frac{1}{5}^\circ$.

Pages 24, 25.

(1) (*b*) 250 pounds.

(2) (*a*) 8 ft. per second; (*b*) 20 ft.

(3) (*a*) 825 pounds; (*b*) $1\frac{1}{4}$ miles; (*c*) about 1000 pounds.

(4) 4·9 sec.

(5) It would be greater by $\frac{1}{180}$ oz.

(6) $j=(m_1 \sin\theta_1 - m_2 \sin\theta_2 - \mu_1 m_1 \cos\theta_1 + \mu_2 m_2 \cos\theta_2)g/(m_1+m_2)$; $T=(\sin\theta_1 + \sin\theta_2 - \mu_1 \cos\theta_1 + \mu_2 \cos\theta_2) m_1 m_2 g/(m_1+m_2)$.

(7) $j = 5·4$ ft. per second; $T=\frac{5}{6}$ pound.

(9) 0·0363.

(10) 0·025.

(11) (*a*) 1528 pounds; (*b*) 3819 pounds.

(12) 589 ft.

Pages 33, 34.

(1) (*a*) 15 270 foot-pounds; (*b*) $30\frac{1}{2}$ ft.

(2) (*a*) 917 pounds; (*b*) 1557 pounds; (*c*) 640 pounds.

(3) (*a*) 1267 foot-tons; (*b*) 4435 foot-tons; (*c*) 5 : 2.

(4) 2016 foot-tons.

(5) 864 ft. per second.

(7) (*a*) $v^2 = 2\,\kappa\rho \log \frac{s_0(s+l)}{s(s_0+l)}$, $v_1 = \infty$; (*b*) $v^2 = 2\,\kappa\rho \log \frac{s_0}{s}$.

(8) $v^2 = 2\,\kappa M\left(\frac{1}{\sqrt{a^2+s^2}} - \frac{1}{\sqrt{a^2+s_0^2}}\right)$.

(9) At the time t, let s be the distance of m_1 and m_2; s_1, s_2 their distances from their initial positions, so that $s_1+s+s_2=s_0$. Then we find $s_1 = \frac{m_2}{m_1+m_2}(s_0-s)$, $s_2 = \frac{m_1}{m_1+m_2}(s_0-s)$, and $\left(\frac{ds}{dt}\right)^2 = 2\,\kappa(m_1+m_2)\left(\frac{1}{s}-\frac{1}{s_0}\right)$. To integrate, put $s=s_0\cos^2\frac{1}{2}\phi$; then we find $t=\frac{s_0}{2}\sqrt{\frac{s_0}{2\,\kappa(m_1+m_2)}}(\phi+\sin\phi)$. The particles meet at the time $t_1 = \frac{\pi s_0}{2}\sqrt{\frac{s_0}{2\,\kappa(m_1+m_2)}}$.

(11) (*a*) $2\pi\sqrt{\frac{e}{g}}$; (*b*) $2\sqrt{\frac{e}{g}}\left[\cos^{-1}\frac{e}{s_0-l_1} + \sqrt{\left(\frac{s_0-l_1}{e}\right)^2 - 1}\right]$.

(12) $s_0 - l_1 = \sqrt{e(2h+e)}$.

(14) The equation $\dfrac{d^2x}{dt^2} = -\mu^2 x - 2\kappa \dfrac{dx}{dt}$ gives: (*a*) when $\kappa^2 > \mu^2$, $x = e^{-\kappa t}(C_1 e^{\sqrt{\kappa^2-\mu^2}\,t} + C_2 e^{-\sqrt{\kappa^2-\mu^2}\,t})$, from which it can be shown that the particle approaches the centre asymptotically, reaching it only in an infinite time; if C_1 and C_2 have opposite signs, the particle will do so after first reaching a maximum elongation and then returning. (*b*) When $\kappa^2 = \mu^2$, $x = e^{-\kappa t}(C_1 + C_2 t)$. (*c*) When $\kappa^2 < \mu^2$, $x = C_1 e^{-\kappa t} \sin(\sqrt{\mu^2 - \kappa^2}\,t + C_2)$, and the particle performs oscillations of period $2\pi/\sqrt{\mu^2-\kappa^2}$ and of decreasing amplitude $C_1 e^{-\kappa t}$.

Pages 36, 37.

(1) 1 watt = 0·73737 foot-pounds per second = 0·001 341 H.P., 1 H.P. = 745·9 watts.

(2) 1 metric H.P. = 735·75 watts = 0·9863 British H.P.

(3) $27\frac{3}{4}$ H.P.

(4) $49\frac{1}{2}$ H.P.

(5) (*a*) 64; (*b*) 224 384.

(6) nearly 200 gallons.

(7) 8789 gallons.

(8) 1 hour.

(9) (*b*) 40 hours; (*c*) about 10 weeks.

Pages 46, 47.

(2) Taking the axis of z vertically upwards, $U = U_0 - mg(z - z_0)$; the equipotential surfaces are the horizontal planes $z =$ const.; the potential energy is $V = mg(z - z_0)$.

(4) Taking the fixed line as axis of z, $U = -\int f(r)\,dr$; the equipotential surfaces are circular cylinders about the axis of z.

(5) Let $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ be the distance of the moving point (x, y, z) from the fixed centre (x_0, y_0, z_0); then the direction cosines of the central force $F = f(r)$ are

$$\alpha = \pm\frac{x-x_0}{r} = \pm\frac{\partial r}{\partial x},\quad \beta = \pm\frac{y-y_0}{r} = \pm\frac{\partial r}{\partial y},\quad \gamma = \pm\frac{z-z_0}{r} = \pm\frac{\partial r}{\partial z},$$

where the + sign holds for a repulsion, the − sign for an attraction; hence $X = \pm f(r)\dfrac{\partial r}{\partial x}$, $Y = \pm f(r)\dfrac{\partial r}{\partial y}$, $Z = \pm f(r)\dfrac{\partial r}{\partial z}$, or, putting $\int f(r)\,dr = F(r)$:

$$X = \pm\partial F(r)/\partial x,\quad Y = \pm\partial F(r)/\partial y,\quad Z = \pm\partial F(r)/\partial z;$$

and finally $U = \pm F(r)$.

(6) $U = \Sigma \int f(r)\,dr$.

(9) Compare J. C. MAXWELL, *Electricity and magnetism*, Vol. I., Arts. 118–123, and the plates I.–III.

(10) Taking the axis of z vertically upwards and denoting by r_1, r_2 the distances from C_1, C_2, we have by the principle of kinetic energy $d(\frac{1}{2}mv^2) = -mgdz - \kappa m dr_1/r_1^2 + \kappa' m r_2 dr_2$; hence the equipotential surfaces are $(c + gz - \frac{1}{2}\kappa' r_2^2)^2 r_1^2 = \kappa^2$. If $\kappa' = 0$, the equation becomes for C_1 as origin $(x^2 + y^2 + z^2)(c + gz)^2 = \kappa^2$; if $\kappa = 0$, we find for C_2 as origin the concentric spheres $x^2 + y^2 + \left(z - \frac{g}{\kappa'^2}\right)^2 = \frac{2c\kappa'^2 + g^2}{\kappa'^4}$.

Pages 52, 53.

(1) $(y_0\dot{z}_0 - z_0\dot{y}_0)x + (z_0\dot{x}_0 - x_0\dot{z}_0)y + (x_0\dot{y}_0 - y_0\dot{x}_0)z = 0$.

(2) $F_i = \kappa m m_i r_i$ has the x-component $X_i = -\kappa m m_i r_i \cdot (x - x_i)/r_i = -\kappa m m_i (x - x_i)$; hence $\Sigma X = -\kappa m \Sigma m_i (x - x_i)$, $\Sigma Y = -\kappa m \Sigma m_i (y - y_i)$, $\Sigma Z = -\kappa m \Sigma m_i (z - z_i)$. Equating these to zero, the position of equilibrium is found as the centroid

$$\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i}, \quad \bar{y} = \frac{\Sigma m_i y_i}{\Sigma m_i}, \quad \bar{z} = \frac{\Sigma m_i z_i}{\Sigma m_i}$$

of the masses m_i. Taking this point as origin, the equation of the plane of motion is the same as in Ex. (1); and the resultant force has the components $-\kappa m \Sigma m_i \cdot x$, $-\kappa m \Sigma m_i \cdot y$, $-\kappa m \Sigma m_i \cdot z$.

Pages 65, 66.

(2) The equation of the orbit given in Ex. (1) is satisfied not only by (x_0, y_0), but also by $(\dot{x}_0/\kappa, \dot{y}_0/\kappa)$; *i.e.* the orbit passes not only through the initial point P_0, but also through the point Q, which is the extremity of the radius vector $OQ = v_0/\kappa$ parallel to v_0; OP_0 and OQ are the conjugate semi-diameters whose equations are $x_0 y = y_0 x$, $\dot{x}_0 y = \dot{y}_0 x$.

(4) The problem reduces to that of constructing the axes of a conic from a pair of conjugate diameters.

(8) The equation of a central conic can be written in the form

$$\frac{1}{p^2} = \frac{1}{a^2} \pm \frac{1}{b^2} \mp \frac{1}{a^2 b^2} r^2,$$

where p is the perpendicular from the centre to the tangent; the upper sign gives the ellipse, the lower the hyperbola. Apply (11), Art. 116.

(9) (*a*) Ellipse; (*b*) hyperbola; (*c*) parabola.

(10) Parabola: $x - x_0 = \frac{\dot{x}_0}{y_0}(y - y_0) - \frac{\kappa m' a}{2 y_0^2}(y - y_0)^2$, where $2a$ is the distance of O_3 from the point O that bisects $O_1 O_2$; the point midway between O and O_3 is taken as origin, and OO_3 as axis of x.

(11) $t = \frac{1}{\kappa}\tan^{-1}\left(\frac{a}{b}\tan\theta\right)$.

Pages 76, 77.

(1) (*a*) $f(r) = \frac{c^2}{a^2 b^2} \cdot r$; (*b*) $f(r) = \frac{c^2 a}{b^2} \cdot \frac{1}{r^2}$.

(2) $v_0 = \sqrt{\mu / r_0}$.

(4) 687 days.

(5) By (24), Art. 138, $v^2 = \frac{2\mu}{r} \mp \frac{\mu}{a}$; as the velocity is not changed instantaneously, we must have $\frac{2\mu}{r} \mp \frac{\mu}{a} = \frac{2\mu'}{r} \mp \frac{\mu'}{a'}$, whence the new major semi-axis a' can be found.

(6) An ellipse with the end of its minor axis at the point where the change takes place.

(7) (*a*) Ellipse with $a = \frac{2}{3} r$; (*b*) parabola.

(8) Differentiate (24), Art. 138, with respect to μ and a.

(9) The periodic time T would be diminished by $\frac{2}{m} T$.

(10) $r = \frac{l}{1 + e\cos\theta}$; hence $x = \frac{l\sin\theta}{1 + e\cos\theta}$, $y = \frac{l\sin\theta}{1 + e\cos\theta}$; differentiating and remembering that $r^2 d\theta/dt = c$, we find

$$\frac{dx}{dt} = -\frac{c}{l}\sin\theta, \quad \frac{dy}{dt} = \frac{c}{l}(\cos\theta + e);$$

eliminating θ, we find the equation of the hodograph

$$x^2 + \left(y - \frac{ec}{l}\right)^2 = \left(\frac{c}{l}\right)^2, \text{ or since } c = \sqrt{\mu l},\ x^2 + \left(y - \frac{\mu e}{c}\right)^2 = \left(\frac{\mu}{c}\right)^2.$$

(11) 1.016 914.

(12) $t = \sqrt{\frac{2a^3}{\mu}}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right)$.

Page 79.

(1) $f(r) = c^2\left[\frac{(n+2)b}{r^{n+3}} + \frac{(n+1)(a^2-b^2)}{r^{2n+3}}\right]$. For $n = -1$, $f(r) = \frac{c^2}{1/b} \cdot \frac{1}{r^2}$; for $n = -2$, $f(r) = c^2(b^2 - a^2) \cdot r$; for $n = 1$ and $a = b$, $f(r) = \frac{3\,bc^2}{r^4}$; for $n = 2$ and $b = 0$, $f(r) = \frac{3\,a^2c^2}{r^7}$.

(2) Taking the diameter perpendicular to the force as axis of x, we find $F = -\frac{a^2c^2}{y^3}$, where a is the radius, c the x-component of the initial velocity.

(3) (*a*) $\frac{c^2}{r^3}\left(\frac{2\,a^2}{r^2} + 1\right)$; (*b*) $\frac{e^2}{r^3}$; (*c*) $\frac{c^2(1+\alpha^2)}{r^3}$; (*d*) $\frac{c^2}{r^3}\left(\frac{2\,n^2a^2}{r^2} + 1 - n^2\right)$.

(4) $\frac{8\,a^2c^2}{r^5}$.

(5) Ellipse, parabola, or hyperbola, according as $\mu \lesseqgtr y_0^2\dot{y}_0^2$, where y_0 is the initial distance from the plane, $\dot{y}_0$ the initial velocity perpendicular to the plane.

(6) $f(r) = -\frac{b^4}{a^2}\frac{\dot{x}_0^2}{y^3}$.

Page 83.

(2) Let ρ_1, ρ_2 be the distances of m_1, m_2 from the common centroid at any time t; $\xi_1 = x_1 - \bar{x}$, etc.; then the equations of the relative motion are

$$\frac{d^2\xi_1}{dt^2} = m_2 f\left(\frac{m_1 + m_2}{m_2}\rho_1\right) \cdot \frac{\xi_1}{\rho_1}, \text{ etc.}; \quad \frac{d^2\xi_2}{dt^2} = m_1 f\left(\frac{m_1 + m_2}{m_1}\rho_2\right) \cdot \frac{\xi_2}{\rho_2}, \text{ etc.}$$

Pages 93, 94.

(4) (*a*) 7½ pounds; (*b*) 735·6 pounds; (*c*) 120 ft. per sec.

(5) 4840 pounds.

(6) $e = \frac{7}{4}\frac{v^2}{R}$ inches.

(7) 8°·6.

(9) 32·20 ft. per sec.

(12) 4·7 pounds.

(13) (*a*) 76 per min.; (*b*) 108.

(14) In Fig. 23, $CD : RF' = PC : PR$, hence $CD = \frac{mg \cdot a}{\omega^2 a} = \text{const.}$

Pages 96, 97.

(1) The integration gives $\tan \frac{1}{4}(\pi + \theta) = \tan \frac{1}{4}(\pi + \theta_0) \cdot e^{\sqrt{\frac{g}{l}}t}$, which gives $t = \infty$ for $\theta = \pi$.

(5) The particle will not leave the circle if N remains positive; $N = 0$ if $\cos \theta_1 = -\frac{2}{3}\frac{h}{l}$.

(6) Greater than 17·94 ft. per sec.

(8) The tangential force is $mg \sin \theta$, if θ is the inclination of the tangent to the horizon; as $\sin \theta = dz/ds$, we have

$$\frac{dv}{dt} = g\frac{dz}{ds}, \text{ or } ds\frac{d^2s}{dt^2} = gdz,$$

whence $\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = g(z - z_0)$. The result also follows from the equation of kinetic energy, since the force-function is $U = mgz + C$.

(9) The equation of motion is the same as (8), Art. 175, except that g is replaced by $g \sin \alpha$.

(10) The particle performs oscillations of period $2\pi/\mu$ if $F = \mu^2 r$.

Page 98.

Taking θ for q, we find $Q = -(mg + \mu\kappa\theta)\kappa$; hence the integration of (17) gives $v^2 = h - 2\kappa g\theta - \frac{\mu\kappa^2}{m}\theta^2$. On the other hand, (14) gives $v^2 = (a^2 + \kappa^2)\left(\frac{d\theta}{dt}\right)^2$. Hence

$$\frac{d\theta}{\sqrt{h - 2\kappa g\theta - \frac{\mu\kappa^2}{m}\theta^2}} = \frac{dt}{\sqrt{a^2 + \kappa^2}}.$$

Pages 102, 103.

(3) (*a*) 9·8 in.; (*b*) 1·23 oz.

(4) As x and y are small of the first order, z differs from r, by (5), only by a small quantity of the second order; hence $z = 0$, $z/r = 1$, so that the third of the equations (6) gives $N = mg$; hence $\ddot{x} = -gx/r$, $\ddot{y} = -gy/r$. Integrating and putting $\sqrt{g/r} = \mu$, we find $x = C_1 \sin \mu t + C_2 \cos \mu t$, $y = D_1 \sin \mu t + D_2 \cos \mu t$. Solving for sin and cos, squaring and adding, we find an ellipse as the required path, just as in Art. 121.

Page 112.

(1) The differential equation is $\ddot{r} - \omega^2 r + g \sin \omega t = 0$. If $r = r_0$, $dr/dt = v_0$ at the time $t = 0$, we find

$$2\,\omega r = \left(\omega r_0 + v_0 - \frac{g}{\omega}\right)e^{\omega t} + \left(\omega r_0 - v_0 + \frac{g}{\omega}\right)e^{-\omega t} + \frac{g}{\omega}\sin \omega t.$$

(2) Let θ be the angle, at the time t, between the radius CP, drawn from the centre C of the circle to the particle P, and the diameter OCA through the fixed point O. Then, taking O as origin, and the initial position OC_0A_0 of the diameter OCA (when P is at A_0) as axis of x, we find

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0, \text{ whence } \left(\frac{d\theta}{dt}\right)^2 = 2\,\omega^2 \cos\theta + C.$$

As the absolute initial velocity is zero, we find $\dot{\theta}_0 = 2\,\omega$. Hence, $\dot{\theta}^2 = 4\,\omega^2 \cos^2 \tfrac{1}{2}\theta$, and finally, $\sin \tfrac{1}{2}\theta = \dfrac{e^{\omega t} - e^{-\omega t}}{e^{\omega t} + e^{-\omega t}}$, or $\tan \tfrac{1}{4}(\pi + \theta) = e^{\omega t}$

(3) Let $x^2 + y^2 = a^2(1 + \alpha t)^2$ be the equation of the circle, whence $x\delta x + y\delta y = 0$; the equation of motion is $\ddot{x}\delta x + \ddot{y}\delta y = 0$; eliminating δx, δy, and integrating, we find $x\dot{y} - y\dot{x} = av_0$. The equation of the circle gives $x\dot{x} + y^{\cdot} = a^2\alpha(1 + \alpha t)$; hence,

$$a(1 + \alpha t)^2\dot{x} = a\alpha(1 + \alpha t)x - v_0\sqrt{a^2(1 + \alpha t)^2 - x^2}.$$

To integrate, put $1 + \alpha t = \alpha\tau$, and then put $x = \xi\tau$. The result is

$$x = a(1 + \alpha t)\cos\frac{v_0 t}{a(1 + \alpha t)}, \quad y = a(1 + \alpha t)\sin\frac{v_0 t}{a(1 + \alpha t)},$$

$$v^2 = a^2\alpha^2 + \frac{v_0^2}{(1 + \alpha t)^2}.$$

Pages 134, 135.

The square of the radius of inertia is:

(1) $\frac{1}{3}l^2$.

(2) (a) $\frac{1}{3}l^2$; (b) $\frac{1}{3}h^2$; (c) $\frac{1}{12}l^2$; (d) $\frac{1}{12}h^2$.

(3) $\frac{1}{2}h^2$.

(4) $\frac{1}{12}a^2$.

(5) $\frac{5}{24}a^2$.

(6) $\frac{1}{3}h^2$.

(7) $\dfrac{n + 1}{n + 3}$

(8) $\frac{1}{4}a^2$.

(9) $\frac{1}{2}a^2$.

(10) $\frac{1}{5}a^2$.

(11) $\frac{1}{5}a^2$, $\frac{1}{5}b^2$, $\frac{1}{5}c^2$.

(12) $\frac{1}{4}(a_1^2 + a_2^2)$.

(13) (*a*) $I = \frac{2}{3}[b^3\alpha + (a-\alpha)\beta^3]$; (*b*) $I = \frac{2}{8}[a^3\beta + (b-\beta)\alpha^3]$.

(14) $I = \frac{1}{24}\rho\delta\,[bh^3 - (b - 2a)(h - 2a)^3]$.

Pages 137, 138.

The square of the radius of inertia is:

(1) $\frac{1}{12}(h^2 + l^2)$.

(2) (*a*) $\frac{1}{24}a^2$; (*b*) $\frac{1}{24}a^2$; (*c*) $\frac{1}{12}a^2$.

(3) (*a*) $\frac{5}{4}a^2$; (*b*) $\frac{1}{2}a^2$; $\frac{3}{2}a^2$.

(4) $\frac{2}{5}a^2$.

(5) $\frac{1}{2}(a_1^2 + a_2^2)$; in the limit, $I = Ma^2$.

(6) Differentiating the moment of inertia in Ex. (4), we find $I = M \cdot \frac{2}{3}a^2$.

(7) (*a*) $\frac{1}{2}a^2$; (*b*) $\frac{3}{2}a^2$; (*c*) $\frac{1}{4}a^2 + \frac{1}{3}h^2$.

(9) (*a*) $\frac{1}{4}b^2$; (*b*) $\frac{1}{4}a^2$; (*c*) $\frac{1}{4}(a^2 + b^2)$.

(10) $\frac{1}{5}(b^2 + c^2)$; $\frac{1}{5}(c^2 + a^2)$; $\frac{1}{5}(a^2 + b^2)$.

(11) $I = \frac{2}{3}[a^3\beta + (b-\beta)\alpha^3] - \frac{1}{2}\dfrac{(a^2\beta + b\alpha^2 - \alpha^2\beta)^2}{a\beta + b\alpha - \alpha\beta}$.

(12) For axis parallel to b, $I = \frac{1}{2}\delta[\frac{1}{3}(h^3 + b\delta^2) + b(h+\delta)^2]$; for axis parallel to h, $I = \frac{2}{3}\delta(2b^3 + h\delta^2)$; for perpendicular axis,

$$I = \tfrac{1}{6}\delta[h^3 + 8b^3 + 3bh^2 + 6bh\delta + 4(b+h)\delta^2].$$

(13) $\frac{1}{2}a^2$.

Pages 149, 150.

(1) The centroidal principal axes are perpendicular to the faces. The moments for these axes are $\frac{1}{3}M(b^2+c^2)$, $\frac{1}{3}M(c^2+a^2)$, $\frac{1}{3}M(a^2+b^2)$. The central ellipsoid is $(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b^2)z^2 = e^4$. For an edge $2a$, $I = \frac{4}{3}M(b^2+c^2)$; for a diagonal $I = \frac{2}{3}M(b^2c^2 + c^2a^2 + a^2b^2)/(a^2+b^2+c^2)$.

For the cube the central ellipsoid becomes a sphere of radius $\frac{1}{3}\sqrt{6}\,a$; for an edge of the cube, $I = \frac{64}{3}a^5$.

(2) Central ellipsoid: $(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b^2)z^2 = 5\epsilon^4$; for l, $q^2 = \frac{1}{5}(6a^2 + b^2)$.

(3) Take the vertex as origin, the axis of the cone as axis of x; then $I_1 = \frac{3}{10}Ma^2$; I_1', *i.e.* the moment of inertia for the yz-plane, $= \frac{3}{5}Mh^2$. As for a solid of revolution about the axis of x $B' = C'$ and $B = C$, we

have $I_2' = I_3' = \frac{1}{2} I_1$, and $I_2 = I_3 = I_1' + \frac{1}{2} I_1$. Hence, $I_2 = I_3 = \frac{3}{5} M (h^2 + \frac{1}{4} a^2)$. At the centroid the squares of the principal radii are $\frac{3}{10} a^2$, $\frac{3}{80} (4 a^2 + h^2)$.

(4) $A = B = C = \frac{2}{3} Ma^2$, $D = E = F = \frac{1}{4} Ma^2$; hence momental ellipsoid: $4 (x^2 + y^2 + z^2) - 3 (yz + zx + xy) = 6 \frac{\epsilon^4}{a^2}$; squares of principal radii: $\frac{1}{6} a^2$, $\frac{11}{12} a^2$, $\frac{11}{12} a^2$.

(5) $q^2 = \frac{1}{2} a^2 (1 + \sin^2 \alpha)$.

(6) $I = \frac{1}{10} \rho \pi a^4 \left(\frac{8}{3} a + H - h + 9 \frac{h^5}{H} \right)$; for $h = a = \frac{1}{3} H$, $q^2 = \frac{27}{106} a^2$.

(7) $A = I_1$, $B = I_2 + M x_1^2$, $C = I_3 + M x_1^2$.

(8) The centroid may be such a point; or if the central ellipsoid be an oblate spheroid, the two points on the axis of revolution at the distance $\pm \sqrt{(I_1 - I_2)/M}$ from the centroid.

(9) The ellipsoid must have the same central ellipsoid as the given body; its equation is $\frac{x^2}{A'} + \frac{y^2}{B'} + \frac{z^2}{C'} = \frac{5}{M}$, where M is the mass, and A', B', C' are the moments of inertia for the principal *planes* of the body at the centroid.

Pages 163, 164.

(1) $\pi \sqrt{\frac{3 a^2 + 4 L^2}{6 g L}}$.

(2) $\frac{2}{3} \sqrt{2}\, a$.

(3) $\frac{25}{9} M \pi^2 r^2$.

(4) $\frac{50}{9} M \pi^2 r^2$.

(5) 375 000 foot-pounds.

(6) $\frac{\pi N^2}{3600 \mu g} \cdot \frac{r^2}{r'}$, where r and r' are expressed in feet.

(7) 4 m. 22 s.

(8) $\sqrt{\frac{3g}{l}}$.

Pages 203–207.

(1) $c = \frac{1}{3} a$, $\alpha = \frac{1}{2} \pi$, $\omega = F/Ma$.

(3) $\frac{5\sqrt{3}}{24} a$.

(8) The body begins to turn about an axis through the fixed point, parallel to the instantaneous axis before impact, with angular velocity $\omega' = \frac{F}{M} \frac{x - x_1}{q^2 + x^2}$.

A LABORATORY MANUAL

OF

PHYSICS AND APPLIED ELECTRICITY.

ARRANGED AND EDITED BY

EDWARD L. NICHOLS,

Professor of Physics in Cornell University.

IN TWO VOLUMES.

VOL. I. Cloth. $3.00.

JUNIOR COURSE IN GENERAL PHYSICS.

BY

ERNEST MERRITT and FREDERICK J. ROGERS.

VOL. II. Cloth. pp. 444. $3.25.

SENIOR COURSES AND OUTLINE OF ADVANCED WORK.

BY

GEORGE S. MOLER, FREDERICK BEDELL, HOMER J. HOTCHKISS,
CHARLES P. MATTHEWS, and THE EDITOR.

The first volume, intended for beginners, affords explicit directions adapted to a modern laboratory, together with demonstrations and elementary statements of principles. It is assumed that the student possesses some knowledge of analytical geometry and of the calculus. In the second volume more is left to the individual effort and to the maturer intelligence of the practicant.

A large proportion of the students for whom primarily this Manual is intended, are preparing to become engineers, and especial attention has been devoted to the needs of that class of readers. In Vol. II., especially, a considerable amount of work in applied electricity, in photometry, and in heat has been introduced.

MACMILLAN & CO.,
66 FIFTH AVENUE, NEW YORK.